Pricing for Coordination

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January 17, 2026

Abstract

A monopolist sells a good to a set of buyers under incomplete information and network externalities. Purchasing decisions being complementary, a pricing policy can yield equilibrium multiplicity. We study how personalized pricing can be used to mitigate this strategic uncertainty, guaranteeing high revenue. An optimal policy takes the form of a posted price with dispersed discounts: the discounts successively rule out low-demand equilibria, while the posted price extracts revenue from the coordinated higher demand. The resulting price dispersion and exclusion patterns are non-assortative, contrasting with various benchmarks we examine. We show how the solution changes with the strength of externalities.

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We thank Ale Bonatti, Ben Brooks, Ken Davidson, Drew Fudenberg, Tan Gan, Ben Golub, Navin Kartik, Ilan Kremer, Hongcheng Li, Stephen Morris, Refine.ink, Wilhelm Schlag, Ilya Segal, Rani Spiegler, Mike Whinston, and various seminar and conference audiences for helpful comments. Simon Boutin and Ferdi Pieroth provided excellent research assistance.

1. Introduction

We study a seller who sells a good to a population of buyers, with two key features. First, there is incomplete information: a buyer's value from purchasing the good depends on some privately known characteristic, namely his type. Second, there are network externalities: a buyer's value from purchasing the good increases with the number of other buyers who also purchase it.

There are many applications for this canonical setting. Incomplete information is the quintessential feature of the monopoly problem, and network externalities are prevalent across industries. Take, for example, a multiplayer game or social media platform. The utility a consumer derives from joining the platform depends on his privately known propensity to play online games or seek social interactions, and also increases with the number of users he can reach via the platform. Similar considerations apply to sellers of file-sharing services, dating websites, and payment apps, among others. In finance, these features are central to a firm raising capital. An investor's incentive to invest with the firm depends on his other planned investments, which are his private information, and is higher if more other investors invest, as the firm is then more likely to be successful.

The seller's problem is to offer each buyer a price to maximize revenue. Prices can be personalized—e.g., via discounts and promotional deals directed to different buyers—but they cannot condition on buyers' types, which are hidden. Buyers decide whether to purchase given the seller's price offers and their types, and given their expectations of other buyers' purchasing decisions. Due to the externalities in consumption, a pricing policy can yield multiple outcomes, with a high total quantity of trade if buyers anticipate that many others will purchase, or a low total quantity if buyers are less optimistic about others' purchases. Low-quantity outcomes are naturally bad for revenue.

Our main result characterizes the optimal pricing policy that guarantees the seller a high revenue, i.e. that maximizes revenue in her worst-case outcome. This policy takes the form of a posted price with dispersed discounts. The seller offers personalized discounts to (some) buyers to successively insulate against low-demand outcomes, and posts a high price to extract revenue from the induced higher demand. The result is price dispersion and exclusion patterns that are non-assortative with respect to buyers' types.

In applications like those described above, list prices together with personalized discounts are common practice. Even if this personalization is partly based on customer data, the allocation of discounts is often arbitrary to a significant extent, and so can be the implied set of excluded buyers. In fact, sellers sometimes offer discounts in a random manner; an example is multi-player gacha games that use loot boxes.[1] More broadly, non-fundamental price dispersion arises when sellers offer selective discounts to sub-populations that are not necessarily distinct in terms of buyers' values; an example is firms applying different discounts depending on the specific credit card a buyer uses.[2] Effectively, practices like these result in a distribution of discounts on a posted price which vary non-assortatively across buyers. Our analysis provides a rationale for these policies and sheds light on their comparative statics.

To illustrate the seller's problem and our main results, we next describe an example that is a special case of our model. Suppose there is a unit mass of ex-ante identical buyers with types drawn uniformly from $\Theta = [0, 1]$. The seller offers a price $p_i \in \mathbb{R}_+$ to each buyer i, and then buyers decide simultaneously whether to purchase. If a buyer of type $\theta_i \in \Theta$ purchases at a price p_i and the total quantity demanded (i.e., the total mass of buyers who purchase) is $q \in [0, 1]$, then the buyer's payoff is $\theta_i q - p_i$. The buyer purchases if, given the total quantity he anticipates, this payoff is weakly greater than his payoff from not purchasing, which is 0. Summarizing the seller's price offers by their distribution $\Pi \in \Delta(\mathbb{R}_+)$, a total quantity q is an equilibrium quantity given Π

[1] The so-called "pity system" employed by these games essentially yields a maximum list price and random discounts that accrue to only some of the players (see, e.g., Gan, 2023).

[2] For instance, discounts for file-sharing services at Dropbox depend on whether the buyer has a Visa or a Mastercard. Similarly, numerous merchants use Groupon to provide selective discounts (see `https://www.groupon.com/merchant/working-with-groupon/merchant-success-stories`). Another example concerns the dating app Tinder, which was found to charge users different prices, with much of the differences being unaccounted for by clear explanatory variables (see European Commission, 2024).

if it is the quantity demanded when buyers anticipate it.[3]

Suppose first that the seller sets only one price for all buyers. That is, Π is degenerate, taking the form of a posted price. It is easy to see that this policy yields the seller a revenue guarantee of zero. If the posted price is strictly positive, there is an equilibrium with zero total quantity—no buyer is willing to purchase at such a price if they anticipate that no other buyer will purchase—so the seller's worst-case revenue is zero. Naturally, revenue is also zero if the posted price is 0.

It follows that the seller must offer a price of 0 to some buyers to ensure a positive demand, while offering others a strictly positive price to generate revenue. How about then just setting two prices? The seller can offer a price of 0 to a share $\pi \in (0, 1)$ of the population and some price p to the remaining $1-\pi$ share. Buyers would anticipate that at least π buyers will purchase, so setting $p \in (0, \pi)$ guarantees a positive revenue. In fact, we can verify that an optimal two-price distribution has $\pi \approx p \approx 0.25$, yielding a worst-case equilibrium quantity $q \approx 0.75$ and revenue $R \approx 0.125$. But why two prices and not more? For example, the seller could choose a uniform price distribution on $[0, p]$ for some $p > 0$ (perturbed to add some small mass at price 0). The optimal such distribution has $p \approx 0.41$, yielding a worst-case equilibrium quantity $q \approx 0.71$ and revenue $R \approx 0.126$.

Our results show that the seller's optimal price distribution in this example is uniform, but only up to a mass point at the top. This distribution is given by $\Pi^*(p) = 2p$ for $p \in [0, p^*)$ and $\Pi^*(p) = 1$ for $p > p^*$, where $p^* \approx 0.28$. We provide an illustration in Figure 1. The worst-case equilibrium under Π^* (once a small mass is added at price 0) has total quantity $q^* \approx 0.72$ and revenue $R^* \approx 0.133$.

The shape of the optimal price distribution reflects two goals of the seller. On the one hand, the seller wishes to coordinate buyers to a high-demand equilibrium. We show that the optimal way to achieve coordination is by

[3] Given Π, if buyers anticipate a total quantity q, then the quantity demanded is $D_q(\Pi) = \int \max\{0, 1 - p/q\} \, \mathrm{d}\Pi(p)$. The quantity q is an equilibrium quantity if $D_q(\Pi) = q$.

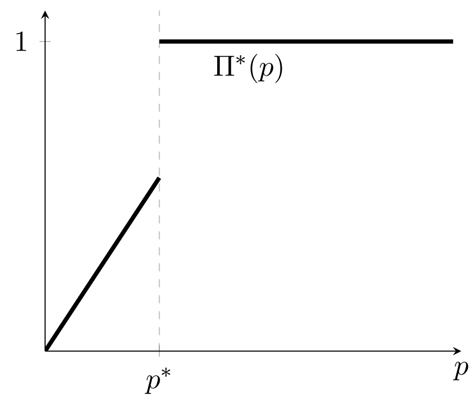

Figure 1: Optimal price distribution in the example described in the Introduction.

using a greedy function that builds demand from the bottom, placing as little mass on low prices as is needed to iteratively rule out low-demand outcomes. This greedy function is uniform in the example above. On the other hand, the seller also wishes to extract revenue given the induced demand. The optimal way to extract revenue under incomplete information is via a posted price, hence the mass point at the highest supported price p^*.

Theorem 1 provides a characterization for our general model. The main primitive of our model is the distribution over buyers' willingness to pay given an anticipated total quantity. We identify concavity conditions on this primitive under which the seller's optimal price distribution is greedy up to its highest supported price, with a mass point at that price. A key point in our proof is that contractions of the price distribution that preserve demand given an anticipated quantity increase both demand and revenue given any higher anticipated quantity. This is why greediness is the optimal form of price dispersion: greedy price distributions are those leaving no room for such contractions. We prove that a greedy function—which corresponds to the solution to an integral equation—is continuous and strictly increasing. Thus, the seller's policy can be interpreted as a posted price with (fully) dispersed discounts.

It is instructive to compare our seller's solution to different benchmarks. A natural benchmark is one without incomplete information, that is, where the seller can condition price offers on both a buyer's identity i and his type θ_i. The problem is then similar to those studied in the literature on contracting with externalities (reviewed below). The seller would optimally induce all buyers to purchase and, under weak conditions, would offer each buyer a different price.[4] Another natural benchmark is one without externalities, that is, where the demand for the seller's good is exogenous. (In the example above, this means that a buyer of type θ_i purchasing at price p_i gets a payoff of $\theta_i \overline{q} - p_i$ for some fixed $\overline{q} \in (0, 1]$, no matter the equilibrium total quantity q.) The problem is then familiar from the literature on monopoly pricing, and yields the celebrated result that a posted price is optimal (Myerson, 1981). In fact, we show that a posted price is also optimal in a benchmark with externalities but without a seller's concern for strategic uncertainty, that is, where the seller maximizes revenue in her best-case rather than worst-case equilibrium.[5]

Our analysis highlights the elements that our seller's solution shares with these benchmarks, as well as the differences. As in the complete-information benchmark, our seller uses price dispersion to achieve coordination; as in the no-externality and best-case benchmarks, she uses a posted price to extract revenue. It is the combination of these features that yields a distinct optimal policy—a posted price with dispersed discounts, as is often seen in applications. Differently from the benchmarks, this policy generates price dispersion and exclusion patterns that are non-assortative: the set of buyers who do not buy in our setting includes both lower and higher types than those who buy. As suggested by our discussion of Theorem 1, these qualitative differences stem from methodological differences in our problem. Unlike in the no-externality and best-case benchmarks, our seller faces constraints due to her worst-case focus, whereas unlike under complete information, simple "divide-and-conquer" strategies are not available to our uninformed seller.

The benchmark comparisons further elucidate the effects of incomplete in-

[4] In the example above, the optimal complete-information price distribution is $\Pi^C(p) = \sqrt{p}$ for $p \in [0, 1]$.

[5] In the example above, the optimal no-externality and best-case posted prices are, respectively, $p^N = \overline{q}/2$ and $p^B \approx 0.22$, yielding total quantities of $q^N = 1/2$ and $q^B \approx 0.66$.

formation and externalities on the total quantity of trade and consumer surplus. Compared to the complete-information benchmark, our seller induces a lower total quantity, while consumer surplus can be higher or lower. The latter possibility suggests a channel through which restricting the seller's access to information can harm consumers. Compared to the no-externality and best-case benchmarks, our seller induces a higher total quantity of trade.[6] Consumer surplus is also higher than in the best-case benchmark, and may be higher than under no externalities. In particular, even though all buyers in our setting may have lower values than under no externalities (e.g., if we take $\overline{q} = 1$ in the no-externality example above), they can still be better off on average due to the higher quantity traded under our seller's policy.

We investigate how the seller's solution varies with the strength of externalities. Say externalities are stronger if buyers' values grow with the anticipated quantity demanded at a higher rate, and are higher for any given anticipated quantity. We find that the stronger the externalities, the less weight the seller's optimal price distribution puts on low prices, and the higher the total quantity that she induces. We also extend our characterization to a population where buyers belong to observable groups of heterogeneous strength of externalities. Similarly to applications where data-based personalization is possible, here discounts can be allocated to buyers based on their group, as well as arbitrarily within each group. We show that the seller offers larger discounts to weak-externality buyers in order to build demand and extract higher revenue from strong-externality buyers.

We conclude with a discussion of variants of our model and potential avenues for future research. In this paper, we focus on a simple monopoly model that combines incomplete information and network externalities. We believe this framework can be enriched in a number of directions—for example, incorporating congestion, dynamics, and two-sided markets—to shed further light on the use of pricing for coordination.

[6] This holds even though the seller extracts lower revenue from any given total quantity under worst-case compared to best-case selection. The finding contrasts with results in complete-information settings; see Segal (2003).

Literature. Our paper relates to three main literatures. First, there is foundational work on monopoly pricing under incomplete information but in the absence of externalities. Key references are Myerson's (1981) seminal study of optimal auction design, and Bulow and Roberts (1989), which relates concepts from Myerson (1981) to the problem of third-degree price discrimination under capacity constraints. We apply their insights directly to our no-externality and best-case benchmarks and build on them in our main analysis.

Second, there is a large literature on markets with network externalities. Classic references include Rohlfs (1974), which highlights the possibility of multiple equilibria under a posted price, and Katz and Shapiro (1985, 1986) and Ellison and Fudenberg (2000), which study models of technology adoption with potentially incompatible products/upgrades. Oren, Smith and Wilson (1982), Csorba (2008), Aoyagi (2013), Veiga (2018), and Markovich, Rayo and Yehezkel (2024) consider settings more similar to ours, but either focus on second-degree price discrimination (with quantity discounts or quality premia) or allow for multilateral schemes that condition on the number of buyers.[7] We are not aware of work in this literature that studies optimal personalized pricing (or more general bilateral contracts; see Section 6) under incomplete information—neither with best-case nor with worst-case selection.[8]

Third, our paper belongs to a growing literature on contracting with externalities that focuses on worst-case selection (or unique implementation). Following respectively the seminal contributions of Segal (2003) and Winter (2004), one strand of this literature studies settings where agents' actions are bilaterally contractible, as in our model, while another strand examines moral hazard problems with unobservable actions. Within the first strand, Halac, Kremer and Winter (2020) consider agents with heterogeneous but observable

[7] Such multilateral schemes are also the focus of Dybvig and Spatt (1983).

[8] Haviv and Winter (2020) examine worst-case implementation schemes in a queue setting, where the seller can charge buyers for priority service with reduced waiting time.

attributes.[9,10] Our main departure is that we study a monopoly setting in which agents' attributes are hidden.[11] We apply insights from this literature when analyzing the complete-information benchmark.

Finally, in addition to these literatures, we relate to papers that predict pricing policies similar to the ones we characterize but in quite different environments. For example, Perry (1984) studies an incumbent firm that seeks to prevent entry and can post different prices for different units of its total supply. The firm uses a continuum of prices, with unlimited supply at the top and just enough supply at each lower price to make entry unattractive. In Heidhues and Kőszegi (2014), a monopolist sells to a loss-averse consumer who forms expectations prior to purchasing based on the monopolist's announced price distribution. To lure the consumer and exploit his attachment, an optimal distribution combines a continuum of sale prices with an atom at a high price. Our paper presents a complementary theory that highlights the role of externalities in consumption. The seller's need to coordinate buyers in the presence of externalities determines the form of price dispersion in our model.

2. Model

Our model introduces strategic complementarities into a canonical monopoly setting with incomplete information. Below we describe the setup, the seller's problem, and our assumptions. We also provide examples of special cases.

[9] Bernstein and Winter (2012) and Sákovics and Steiner (2012) examine related models with observable heterogeneity. More broadly related, Chan (2024) studies different implementation requirements in weighted potential games; Ali, Haghpanah, Lin and Siegel (2022) and Gan and Li (2024) consider worst-case selection for a seller of information facing equilibrium multiplicity due to market expectations; and a number of papers study coordination via exclusionary contracts, including Rasmusen, Ramseyer and Wiley (1991), Innes and Sexton (1994), Segal and Whinston (2000), Spiegler (2000), and Genicot and Ray (2006).

[10] Within the second strand, see, e.g., Eliaz and Spiegler (2015); Moriya and Yamashita (2020); Halac, Lipnowski and Rappoport (2021, 2022); Chan (2023); Cusumano, Gan and Pieroth (2023); Camboni and Porcellacchia (2024); Halac, Kremer and Winter (2024).

[11] Che and Spier (2008) consider a two-agent example with hidden information in their analysis of coordination in settlement offers. In the aforementioned paper by Ali et al. (2022), the seller sets a non-personalized disclosure fee under endogenous incomplete information about buyer values.

2.1. Setup

We study a seller who sells a good to a population of buyers, each with a unit demand. Buyers' identities $i \in I := [0,1]$ are uniformly distributed and independent of their payoff types $\theta \in \Theta := [0,1]$, which have distribution G.[12] The seller makes a price offer $p_i \in \mathbb{R}_+$ to each buyer $(i,\theta) \in I \times \Theta$. Prices are personalized, namely they can depend on a buyer's identity i. The offered price however cannot condition on a buyer's type θ, which is the buyer's private information.[13]

Given the price offers, the buyers simultaneously decide whether or not to purchase from the seller. Denote the total quantity of purchases—i.e., the total mass of buyers who purchase—by $q \in [0,1]$. If a buyer of type $\theta \in \Theta$ purchases at a price p_i and the total purchased quantity is q, the buyer gets a payoff of

$$u(\theta, q) - p_i. \tag{1}$$

The continuous function $u : \Theta \times [0,1] \to \mathbb{R}_+$ is strictly increasing in its second argument for a full-measure set of types, reflecting that buyers' purchasing decisions are complementary. We also assume $u(\theta, q)$ is strictly increasing in θ whenever $q > 0$. If a buyer does not purchase, his payoff is 0.

The random variable $u(\cdot, q)$ represents a buyer's willingness to pay given an anticipated total quantity q. Let $F_q : \mathbb{R} \to [0,1]$ denote its cumulative distribution function. We assume F_q has support $[0, \overline{v}(q)] \subset \mathbb{R}_+$ for $q \in [0,1]$, where $\overline{v}$ is continuously differentiable with $\overline{v}'(q) > 0$ for every $q \in (0,1]$. In particular, the lowest value is zero, whereas the highest value is strictly increasing in anticipated quantity. We further make the "cold-start" assumption that $\overline{v}(0) = 0$, so a buyer's value is almost surely zero if he anticipates no other

[12] We set the type space equal to $[0,1]$ for simplicity and because all of our examples are of this form, but our analysis applies without change for any Polish space Θ. All results continue to hold in that more general setting, with the exception of Corollary 2, which concerns an order on types.

[13] Since a buyer's type is independent of his identity, it is also independent of his price offer. Using this fact, we show in Section 6 that our focus on personalized price offers is without loss of generality within the class of public bilateral contracts.

buyer will purchase.[14] For strictly positive anticipated quantity $q \in (0,1]$, we suppose F_q admits a density f_q which is strictly positive on $(0, \overline{v}(q)]$, and that $f_q(v)$ is continuous in (q,v) where $0 \leq v \leq \overline{v}(q)$, having a partial derivative with respect to q that is also continuous in (q,v) over this domain.

Given a fixed anticipated quantity $q \in [0,1]$, the quantity that buyers demand and the seller's revenue can be easily computed. Assume that a buyer who is indifferent over purchasing chooses to purchase.[15] Then, given anticipated quantity q, the **quantity demanded** from a price p is equal to the mass of buyers whose willingness to pay is weakly greater than p, denoted $D_q(p) := 1 - F_q(p^-)$, and the quantity demanded from a price distribution $\Pi \in \Delta(\mathbb{R}_+)$ is $D_q(\Pi) := \int D_q(p)\ \mathrm{d}\Pi(p)$. Similarly, the seller's **revenue** from a price p is $R_q(p) := pD_q(p)$, and her revenue from a price distribution Π is $R_q(\Pi) := \int R_q(p)\ \mathrm{d}\Pi(p)$.[16]

2.2. Seller's problem

The seller's price offers $(p_i)_{i \in I}$ induce a coordination game between the buyers. In this game, each buyer (i, θ) simultaneously makes a decision of whether to purchase, with his payoff from purchasing given by (1). Since a buyer's identity conveys no information about his type, we can summarize the seller's price offers by their distribution $\Pi \in \Delta(\mathbb{R}_+)$. Given such a price distribution Π, if all buyers anticipate a total quantity q, the total quantity demanded is $D_q(\Pi)$. Thus, a total quantity q is an **equilibrium quantity** given Π if it is the quantity demanded when buyers anticipate it: $D_q(\Pi) = q$.

Due to the complementarity in buyers' purchasing decisions, multiple equilibrium quantities may arise given a price distribution Π. The seller wishes to guarantee a high revenue, and is therefore concerned with maximizing revenue

[14] We relax the zero-lowest-value and cold-start assumptions in Section 6.

[15] Without this tie-breaking assumption, our results would remain unchanged if we allow the seller to use (slightly) negative prices for a small fraction of the buyer population.

[16] In a slight abuse of notation, we let $D_q(\Pi)$ and $R_q(\Pi)$ be similarly defined by such integrals for any function $\Pi : \mathbb{R}_+ \to \mathbb{R}$ such that the integral is well defined.

in the worst-case equilibrium. Formally, her optimal value is given by

$$\sup_{\Pi \in \Delta(\mathbb{R}_+)} \min_{q \in [0,1]} R_q(\Pi) \quad \text{subject to} \quad D_q(\Pi) = q. \tag{P}$$

As a consequence of Lemma 2 in the Appendix, the minimum in program (P) is well-defined for any price distribution Π. However, the supremum is generally not attained. Hence, we say that (Π^*, q^*) is **optimal** if there exists a sequence $(\Pi_k, q_k)_k$ that converges to (Π^*, q^*) such that quantity q_k is the worst-case equilibrium quantity given price distribution Π_k for every k and $R_{q_k}(\Pi_k)$ converges to the seller's optimal value in (P).

Remark 1. The complementarity in buyers' purchasing decisions implies that the seller's revenue R_q is increasing in q. Hence, a worst-case equilibrium and a best-case equilibrium for the seller are respectively a lowest-quantity equilibrium and a highest-quantity equilibrium, and these equilibria exist for a given price distribution (see Lemma 2 in the Appendix).

Remark 2. While we have stated the seller's problem as maximizing revenue in the worst-case equilibrium, in our setting this is equivalent to maximizing revenue in the worst-case rationalizable outcome. The reason is that buyers' purchasing decisions are complementary, and thus the game they play under any price distribution is supermodular. The equivalence then follows from Guesnerie and Jara-Moroni (2011), who extend results of Milgrom and Roberts (1990) to games with a continuum of players. Further building on this observation and given Assumption 2 below, it will also turn out that the seller's worst-case problem is essentially equivalent to a more constrained one that maximizes revenue subject to inducing a unique equilibrium.

Remark 3. We have set up the model as one with positive externalities (i.e., with buyers' payoffs increasing in q). This implies that the worst-case equilibrium for the seller is also the worst-case equilibrium for the buyers. However, virtually nothing in our analysis changes if we assume that a buyer's value from purchasing is 0 while that from not purchasing is $-u(\theta, q)$.[17] In this case, there are negative externalities on nontraders (cf. Segal, 1999), and the seller's worst-case equilibrium is the best-case equilibrium for the buyers.[18]

2.3. Concavity assumptions

We make three assumptions that we maintain throughout our analysis. Observe that while it is natural to describe our model in terms of the buyers' willingness-to-pay function $u(\theta, q)$ and their type distribution G (as we will do when providing examples in Section 2.4), there is a sense in which this is over-specified. In fact, different pairs of u and G map to the same distribution F_q over buyers' willingness to pay and therefore yield the same equilibrium conditions. Thus, we express our assumptions in terms of our primitive F_q.

Our first two assumptions concern the shape of externalities in buyers' purchasing decisions. Our model is one in which externalities are positive (but see Remark 3) and increasing (i.e., purchasing decisions are strategic complements).[19] Our first assumption sharpens the increasing externalities property by requiring a monotone likelihood ratio property (MLRP):

Assumption 1 (MLRP). *For any $0 < q \leq \tilde{q} \leq 1$, the likelihood ratio $f_{\tilde{q}}(v)/f_q(v)$ is weakly increasing in v over $(0, \overline{v}(q)]$.*

Our model assumption that $u(\cdot, q)$ is increasing in q means that the distribution of willingness to pay under an anticipated quantity $\tilde{q}$ first-order stochastically dominates that under any lower anticipated quantity $q \leq \tilde{q}$, and MLRP requires such dominance even when conditioning on any set of values. This property implies that for any quantities $0 < q \leq \tilde{q} \leq 1$, the demand $D_{\tilde{q}}(\cdot)$ is a concave transformation of the demand $D_q(\cdot)$ over the common support $[0, \overline{v}(q)]$.

Our second assumption requires the demand function to be strictly concave in anticipated quantity.

[17] The only results that change in this case are the consumer surplus claims in Section 4.
[18] Hence, the seller's concern that buyers may coordinate on her worst-case outcome becomes equivalent to a concern that buyers may coordinate on their best-case outcome.
[19] This terminology follows Segal (2003).

Assumption 2 (**Concave externalities**). *Wherever* $q \in [0,1]$ *and* $p \in \mathbb{R}_{++}$ *have* $p < \overline{v}(q)$*, the demand function* $D_q(p)$ *is strictly concave in* q*.*

This assumption can be interpreted as saying that externalities in our model are concave: a buyer's probability of purchasing increases with the anticipated total quantity demanded at a decreasing rate.

Finally, for our third assumption, we define the **cross virtual value** associated with a buyer's willingness to pay v. For any $0 < q \leq \tilde{q} \leq 1$, the cross virtual value function $\varphi_{q,\tilde{q}} : (0, \overline{v}(q)] \to \mathbb{R}$ is given by

$$\varphi_{q,\tilde{q}}(v) := \frac{f_{\tilde{q}}(v)}{f_q(v)} \left[v - \frac{1 - F_{\tilde{q}}(v)}{f_{\tilde{q}}(v)} \right].$$

This function is exactly the Myerson virtual value function under total quantity $\tilde{q}$ in the special case that $q = \tilde{q}$, and is otherwise the Myerson virtual value function under $\tilde{q}$ scaled by the likelihood ratio $f_{\tilde{q}}(v)/f_q(v)$. Recall that Myerson regularity says that the virtual value function $\varphi_{\tilde{q},\tilde{q}}$ is increasing. We make an analogous assumption on the cross virtual value function:

Assumption 3 (**Cross regularity**). *For any* $0 < q \leq \tilde{q} \leq 1$*, the cross virtual value function* $\varphi_{q,\tilde{q}}(v)$ *is strictly increasing in* v *over* $(0, \overline{v}(q)]$*.*

For intuition, fix a quantity $\tilde{q}$. As noted by Bulow and Roberts (1989), the Myerson virtual value function corresponds to the seller's marginal revenue, and thus Myerson regularity implies that the seller's revenue $R_{\tilde{q}}(\cdot)$ is concave in the quantity demanded under $D_{\tilde{q}}(\cdot)$. Cross regularity serves an analogous role to regularity, but applies across different anticipated quantities $q \leq \tilde{q}$ in our seller's problem. In particular, when evaluating changes to the price distribution, our seller considers the effect on demand given hypothetical pessimistic buyer beliefs about quantity, as well as the effect on revenue given the actual equilibrium quantity. Cross regularity says that for any anticipated and actual total quantities $0 < q \leq \tilde{q} \leq 1$, the revenue $R_{\tilde{q}}(\cdot)$ is a concave transformation of the demand $D_q(\cdot)$ over the common support $[0, \overline{v}(q)]$.

2.4. Examples

We will illustrate our results with the following special cases of our model.

Linear demand. Suppose a buyer's willingness to pay given type θ and anticipated quantity q is $u(\theta,q)=\theta\overline{v}(q)$ for differentiable $\overline{v}$ satisfying $1/\overline{v}(q)$ strictly convex in q over $(0,1]$ (as well as $\overline{v}(0)=0$ and $\overline{v}'>0$), and let the type distribution G be uniform. The convexity condition on $1/\overline{v}$ is equivalent to our concave externalities assumption; it holds, for example, if $\overline{v}$ is log-concave. The demand function takes the linear form $D_q(p)=1-p/\overline{v}(q)$, and one can verify that all of our model assumptions are satisfied. Our benchmark comparisons in Section 4 and comparative-static results in Section 5 will focus on this environment.

Proportional values. Suppose a buyer's willingness to pay given type θ and anticipated quantity q is $u(\theta,q)=\theta q$, and let g be a continuously differentiable, strictly positive density on $(0,1]$ associated with type distribution G. Then, MLRP and cross regularity say, respectively, that for all $\alpha>1$, the ratio $g(\theta)/g(\alpha\theta)$ is weakly increasing in θ and the expression

$$\frac{g(\theta)}{g(\alpha\theta)}\left[\theta-\frac{1-G(\theta)}{g(\theta)}\right]$$

is strictly increasing in θ wherever $g(\alpha\theta)$ is strictly positive. Concave externalities says that $G(p/q)$ is convex in q, and turns out to be implied by the other conditions for this class of environments. An example that satisfies these conditions is the power distribution with $G(\theta)=\theta^{\kappa}$ for $\kappa\geq 1$.

Other examples. The examples described above take a willingness-to-pay function of the form $u(\theta,q)=\theta\overline{v}(q)$. Our model can also accommodate other formulations; for example, $u(\theta,q)=e^{\theta q}-1$ paired with a uniform type distribution G would satisfy all of our assumptions.

A natural setting that is outside our model as stated is one in which a buyer's willingness to pay takes an additive form, $u(\theta,q)=\theta+q$. This formulation does not satisfy our zero-lowest-value and cold-start assumptions. Both of these assumptions however can be relaxed—see Section 6 for details—and our analysis remains valid in settings like the additive one.

3. Optimal price distribution

Our analysis proceeds as follows. Section 3.1 presents an auxiliary program that elucidates the key constraints in the seller's problem. We use this auxiliary program in Section 3.2 to derive our main result on the seller's optimal price distribution, and we provide intuition for the proof of this result in Section 3.3.

3.1. Which constraints matter?

Recall that (Π^*, q^*) is optimal if it is the limit of a sequence $(\Pi_k, q_k)_k$ of price distributions and corresponding worst-case equilibrium quantities whose revenue $R_{q_k}(\Pi_k)$ converges to the seller's optimal value in (P). We next show that (Π^*, q^*) can be computed as the solution to an auxiliary program.

Proposition 1. (Π^*, q^*) *is optimal if and only if it solves*

$$\max_{\Pi \in \Delta(\mathbb{R}_+),\ q \in [0,1]} R_q(\Pi) \tag{P*}$$

$$\text{subject to} \quad D_{\hat{q}}(\Pi) \geq \hat{q} \quad \forall \hat{q} \in (0, q).$$

Moreover, this program has a maximizer, generating strictly positive revenue.

Observe that program (P*) maximizes over both a price distribution $\Pi \in \Delta(\mathbb{R}_+)$ and an equilibrium quantity $q \in [0,1]$, and any optimum (Π^*, q^*) in (P*) must satisfy the equilibrium condition $D_{q^*}(\Pi^*) = q^*$ (for otherwise raising q^* would yield a strict improvement). However, there are additional constraints that (P*) imposes for (Π^*, q^*) to be optimal given the seller's focus on worst-case outcomes. Plainly, the price distribution cannot admit any lower-quantity equilibrium, from which it follows that the demand $D_{\hat{q}}(\Pi)$ at any anticipated quantity $\hat{q} < q^*$ must exceed $\hat{q}$. Program (P*) imposes these demand constraints as weak inequalities, with the solution being the limit of a sequence $(\Pi_k, q_k)_k$ that satisfies the constraints strictly for every k.

To prove Proposition 1, we first show that the auxiliary program (P^*) is a relaxation of the original program (P). In fact, any price distribution $\Pi \in \Delta(\mathbb{R}_+)$ and its corresponding worst-case equilibrium quantity q are feasible in (P^*): if $q = 0$, the program imposes no constraints, and if $q > 0$, then this being the lowest equilibrium quantity under Π implies that the constraints in (P^*) hold strictly for all $\hat{q} \in [0, q)$. Next, in the other direction, we show that (P^*) cannot yield strictly higher revenue than (P). Given (Π, q) feasible in (P^*), we construct a perturbed price distribution Π_ε which coincides with Π except for a small fraction ε of buyers who are offered a zero price. For every $\varepsilon > 0$, the price distribution Π_ε generates a worst-case equilibrium quantity $q_\varepsilon \geq q$. Hence, since revenue is increasing in the quantity demanded and Π_ε converges to Π as $\varepsilon \to 0$, we obtain $R_{q_\varepsilon}(\Pi_\varepsilon) \geq R_q(\Pi)$ in this limit.

While program (P^*) clarifies which constraints are relevant in the seller's problem, it is not immediate what its solution looks like. The seller chooses a continuum of prices which must satisfy a continuum of demand constraints. At each of these constraints, she faces tradeoffs between increasing one price and lowering another to preserve demand, and the tightness of the constraints depends on the relative slopes of the demand function at different anticipated total quantities. It might be intuitive to think that the solution to (P^*) should satisfy all the constraints with equality—but this may not be feasible for a given target quantity, and even when feasible, we will see that it is not optimal.

In the next sections, we show that a key principle behind the solution to (P^*) is that price dispersion is bad for revenue, so quantity-preserving contractions of the price distribution benefit the seller. This does not mean that a degenerate price distribution is optimal; as discussed in the Introduction, that would yield a revenue guarantee of zero. Our analysis will instead show how this principle can be used to pin down the optimal form of price dispersion.

3.2. Posted price with dispersed discounts

We define a class of functions that we will use in our characterization of the seller's optimal pricing policy.

Definition 1. *Let $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be right-continuous and nondecreasing. Given*

$p \in \mathbb{R}_+$, say Γ is ***greedy up to p*** *if it satisfies*

$$D_{\hat{q}}(\Gamma) = \hat{q}$$

for all $\hat{q} \in (0,1)$ with $\overline{v}(\hat{q}) \le p$. Say Γ is ***greedy*** *if it is greedy up to every $p \ge 0$.*

A function Γ that is greedy up to p satisfies the demand constraints in program (P*) with equality for all anticipated quantities for which the highest willingness to pay is no greater than p.[20] This means that Γ iteratively sets to zero the demand-constraint difference $D_{\hat{q}}(\Gamma) - \hat{q}$ starting from the lowest anticipated quantity up to $\underline{q}(p) := \overline{v}^{-1}(p)$. Intuitively, a greedy function follows a greedy procedure: for each anticipated quantity $\hat{q}$ starting from 0, given a measure over prices $[0, \overline{v}(\hat{q}))$, the seller pushes up the next prices as much as possible subject to satisfying the demand constraint at $\hat{q}$. Following this greedy procedure up to q is equivalent to solving the Volterra integral equation $\int_0^{\overline{v}(\hat{q})} \Gamma(p) f_{\hat{q}}(p) \, \mathrm{d}p = \hat{q}$ for all $\hat{q} \in (0, q)$.

The next theorem presents our main result.

Theorem 1. *Suppose (Π^*, q^*) is optimal, and let p^* be the highest price in the support of Π^*. Then $q^* < 1$, and Π^* is greedy up to $p^* < \overline{v}(q^*)$, with a mass point at p^*.*

A seller's optimal price distribution balances two goals. On the one hand, being concerned with worst-case outcomes, the seller wishes to insulate against low-demand equilibria. She does so by using a greedy function that seeds demand from the bottom, placing as little mass on low prices as is needed to iteratively rule out low-demand outcomes and coordinate buyers to a high-demand equilibrium.[21] On the other hand, the seller also wishes to extract

[20] Note that Γ need not be a distribution function; in particular, it can take values greater than 1.

[21] Recall that while the demand constraints in program (P*) are weak inequalities, the solution (Π^*, q^*) is the limit of a sequence $(\Pi_k, q_k)_k$ which, for every k, satisfies the demand constraints strictly and thus rules out equilibrium quantities $\hat{q} < q_k$.

revenue from the induced high demand. She achieves this via the mass point at the highest offered price p^*. The resulting price distribution minimizes the demand constraints in program (P*) up to anticipated quantity $\underline{q}^* := \underline{q}(p^*)$ and satisfies the constraints with slack for quantities in $(\underline{q}^*, q^*)$.

Remark 4. In many environments, one can verify directly that the seller's problem admits a unique greedy function Γ^* over $[0, \overline{v}(1))$. In such cases, Theorem 1 reduces the problem to a one-parameter optimization over q^*, as any optimal (Π^*, q^*) must then have Π^* coincide with Γ^* up to its highest supported price $p^* \in (0, \overline{v}(q^*))$, and such price p^* is in one-to-one correspondence with quantity q^*. This is always true in the linear demand and proportional values environments—see Lemma 12 and the proof of Corollary 3 in the Appendix.

Theorem 1 suggests an appealing interpretation for the seller's optimal pricing policy: the seller posts a high price and simultaneously offers personalized discounts to some buyers to build a high demand. As discussed in the Introduction, the use of list prices together with promotions and special deals that vary across buyers is common in applications. The shape of Π^* tells us precisely how these personalized discounts are optimally distributed in the population. We show in the Appendix that any greedy function must be continuous and strictly increasing. Hence, it follows from Theorem 1 that the seller's optimal price distribution has only one mass point, and personalized discounts are (fully) dispersed across buyers.

Corollary 1. *Any optimal price distribution is continuous and strictly increasing up to a mass point at the top of its support. Said differently, the seller's optimal policy is a posted price with dispersed discounts.*

The seller offers buyers personalized discounts, but she cannot condition these offers on buyers' hidden types. As a result, the price dispersion she induces is non-assortative: higher types may receive a larger or smaller discount than lower types. Moreover, this means that the induced exclusion pattern is also non-assortative. Let θ^* be defined by $u(\theta^*, q^*) = p^*$. In the seller's solution (Π^*, q^*), buyers of type $\theta \in [\theta^*, 1]$ buy with certainty, as their willingness to pay exceeds the posted price p^* and thus any possibly discounted price they may be offered. But whether a buyer of type $\theta \in (0, \theta^*)$ buys depends on the magnitude of his discount. For any such buyer who is excluded—i.e., (i, θ_i) with $0 < \theta_i < \theta^*$ and $u(\theta_i, q^*) < p_i$—there exist buyers of lower and higher type who are not excluded—i.e., (j, θ_j) and (ℓ, θ_ℓ) with $\theta_j < \theta_i < \theta_\ell$ and with $u(\theta_j, q^*) \geq p_j$, and $u(\theta_\ell, q^*) \geq p_\ell$. We summarize this observation in the following corollary.

Corollary 2. *The seller's optimal policy induces price dispersion and exclusion patterns that are non-assortative with respect to buyers' types.*

As we will see, this non-assortativity arises from the combination of incomplete information and externalities in our seller's problem; it is absent in the benchmark settings we study in Section 4.

Figure 2 illustrates our results with three examples. The first two examples (top and middle graphs) belong to the linear demand environment—with a willingness-to-pay function $u(\theta, q) = \theta \overline{v}(q)$ and a uniform distribution of types. The first example is the one discussed in the Introduction, with $\overline{v}(q) = q$, and the second example takes $\overline{v}(q) = q + q^2$. The third example (bottom graphs) belongs to the proportional values environment, with $u(\theta, q) = \theta q$ and a power distribution of types.

For each example, the graphs on the left depict the unique greedy function $\Gamma^*(p)$ (gray dotted line) and the seller's optimal price distribution $\Pi^*(p)$ (black solid line), which is also unique. Observe that the first and third examples both have proportional values, and while they assume different distributions of types, in both cases the unique greedy function is uniform (and given by $\Gamma^*(p) = p/\mathbb{E}[\theta]$). This is not a coincidence, as we report in the next corollary.

Corollary 3. *In the proportional values environment, the seller's policy is a posted price with uniform discounts.*

The graphs on the right in Figure 2 show the distribution of purchasing buyers induced by the seller's solution (Π^*, q^*). Specifically, we plot the density of buyers' types $g(\theta)$, and shade the area under this curve corresponding to the

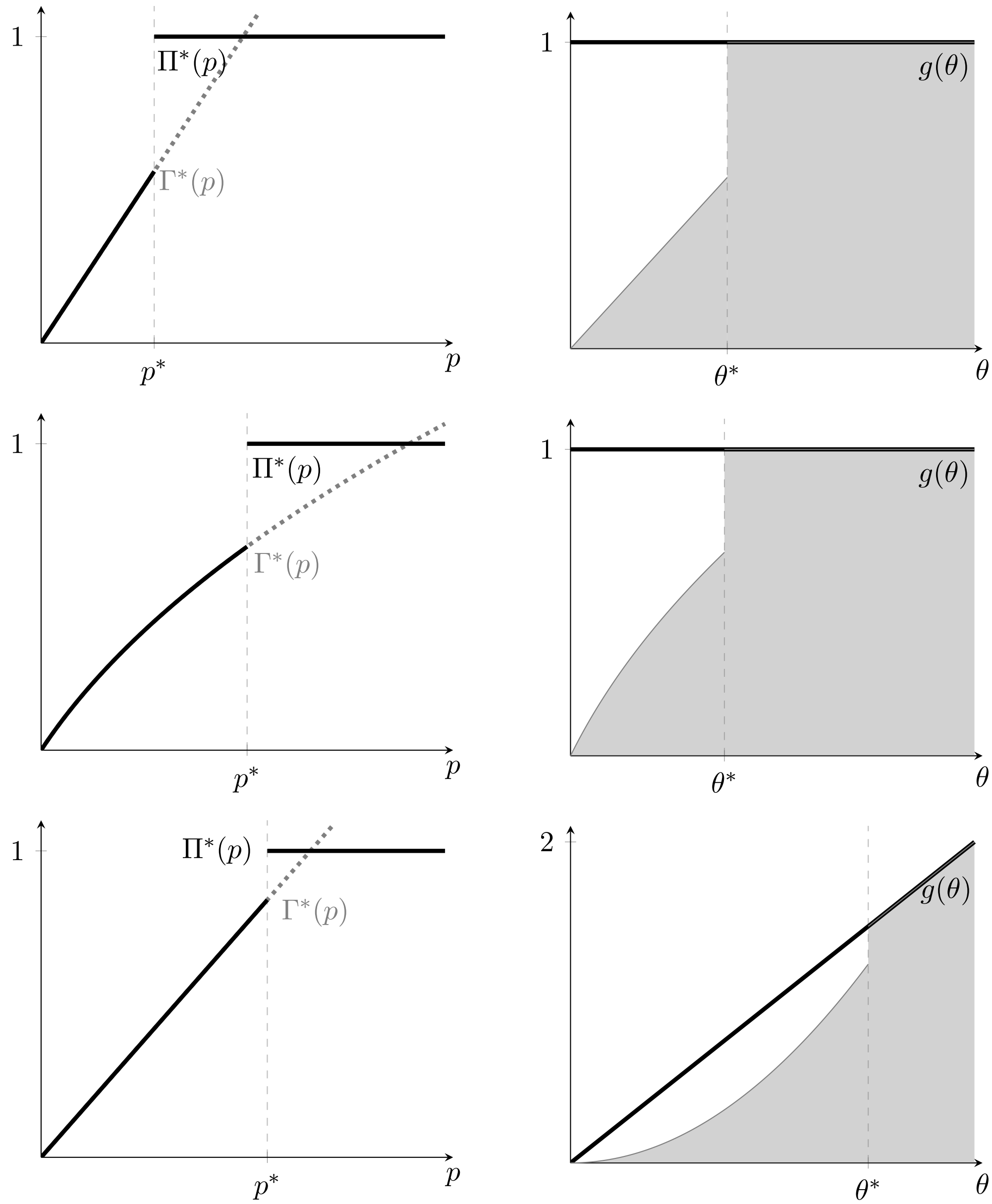

Figure 2: Left: Greedy function (gray dotted line) and optimal price distribution (black solid line). Right: Density of types (black line) with shaded area underneath indicating those purchasing. The top panel takes a linear demand environment with $\overline{v}(q) = q$, and has $\Gamma^*(p) = 2p$ with $p^* \approx 0.28$ and $q^* \approx 0.72$. The middle panel takes a linear demand environment with $\overline{v}(q) = q + q^2$, and has $\Gamma^*(p) = (\sqrt{1+4p}-1)/2 + p/\sqrt{1+4p}$ with $p^* \approx 0.51$ and $q^* \approx 0.76$. The bottom panel takes proportional values with $g(\theta) = 2\theta$, and has $\Gamma^*(p) = (3/2)p$ with $p^* \approx 0.56$ and $q^* \approx 0.76$. The right graphs have the area under $g(\cdot)\Pi^*(u(\cdot, q^*))$ shaded.

mass of buyers who buy (i.e., the entire area under the function $g(\cdot)\Pi^*(u(\cdot, q^*))$. The non-shaded area under $g(\theta)$ represents the excluded buyers. The fact that this area is strictly interior for a range of types $(0, \theta^*)$ implies that exclusion is non-assortative.

3.3. Intuition for proof of Theorem 1

We next provide intuition for the proof of Theorem 1. To highlight the main ideas, we focus on the linear demand environment. We comment on the differences with respect to our general proof at the end of this section.

As a preliminary step, we note that under a linear demand, we can rewrite the demand constraints in the auxiliary program (P*) as follows:

$$\int_0^{\overline{v}(\hat{q})} \Pi(p)\mathrm{d}p \geq \hat{q}\overline{v}(\hat{q}) \quad \forall \hat{q} \in (0, q). \tag{2}$$

One can readily verify that all of these constraints are satisfied with equality if Π agrees with the unique greedy function Γ^* up until at least $\overline{v}(q)$, where, recalling $\underline{q} := \overline{v}^{-1}$,

$$\Gamma^*(p) = \underline{q}(p) + p\underline{q}'(p). \tag{3}$$

Suppose by contradiction that (Π^*, q^*) is optimal and Π^* is not greedy up to its highest supported price p^*. Since we have shown in Proposition 1 that the seller's optimal value is strictly positive, we take $q^*, p^* > 0$. By definition of the greedy function Γ^*, and assuming here that $\Pi^* - \Gamma^*$ is piecewise monotone, it follows that there exists some price $p' \in [0, \overline{v}(q^*))$ such that $\Pi^*(p) = \Gamma^*(p)$ for $p \leq p'$ and $\Pi^*(p) > \Gamma^*(p)$ right above p'. An illustration is provided in the left panel of Figure 3, where we have drawn $\Gamma^*(p)$ (gray dotted line) for the same environment as in the top example of Figure 2.

There are two scenarios to consider. First, suppose that there exists a price $p'' \in (p', \overline{v}(q^*))$ such that

$$\int_{p'}^{p''} [\Pi^*(p) - \Gamma^*(p)]\,\mathrm{d}p = 0. \tag{4}$$

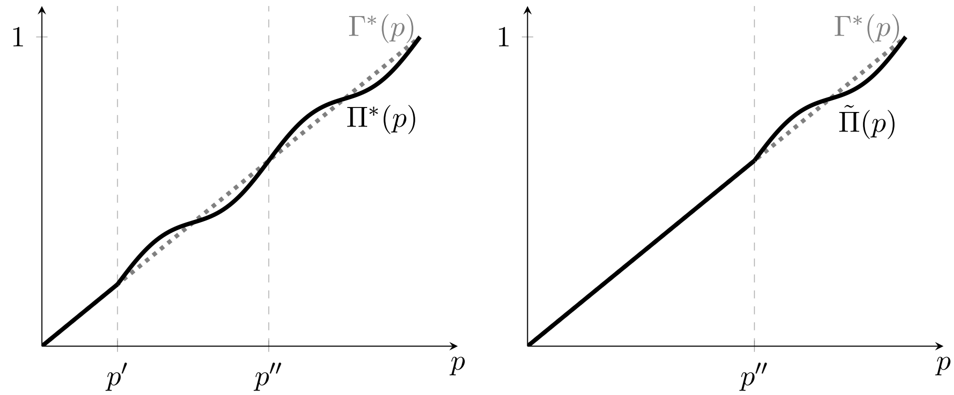

Figure 3: Illustration of arguments in Section 3.3. See the text for details.

Then we can take the lowest such price p'', in which case

$$\int_{p'}^{\hat{p}} [\Pi^*(p) - \Gamma^*(p)] \, \mathrm{d}p > 0 \quad \forall \hat{p} \in (p', p''), \tag{5}$$

as illustrated in Figure 3.

Now let us define a new price distribution $\tilde{\Pi}$ which coincides with the greedy function Γ^* up to p'' and is otherwise equal to Π^*:[22]

$$\tilde{\Pi}(p) = \begin{cases} \Gamma^*(p) & \text{for } p < p'' \\ \Pi^*(p) & \text{otherwise.} \end{cases}$$

The right panel of Figure 3 provides an illustration. By definition of Γ^*, the price distribution $\tilde{\Pi}$ satisfies the demand constraints for all anticipated quantities $\hat{q} \in (0, \underline{q}(p''))$. Moreover, observe that by (4) and (5), $\tilde{\Pi}$ is a mean-preserving contraction of Π^* below p''. Recalling our discussion of MLRP (Assumption 1) as a relative concavity condition, this property therefore implies $D_{\hat{q}}(\tilde{\Pi}) \geq D_{\hat{q}}(\Pi^*)$ for all $\hat{q} \in [\underline{q}(p''), 1]$, which means that $\tilde{\Pi}$ also satisfies the demand constraints for all anticipated quantities $\hat{q} \in [\underline{q}(p''), q^*)$. Furthermore, cross regularity (Assumption 3) implies $R_{\hat{q}}(\tilde{\Pi}) > R_{\hat{q}}(\Pi^*)$ for all $\hat{q} \in [\underline{q}(p''), 1]$. It follows that $\tilde{\Pi}$ yields strictly higher revenue than Π^*, contradicting the

[22] We can verify that the price p'' satisfies $p'' \leq p^*$ and $\Gamma^*(p'') \leq \Pi^*(p'')$, so $\tilde{\Pi}$ is a distribution function.

assumption that Π^* is optimal.

We are then left with the second scenario, in which no $p'' \in (p', \overline{v}(q^*))$ exists that satisfies equation (4). In this case, each $\hat{p} \in (p', \overline{v}(q^*))$ has

$$\int_{p'}^{\hat{p}} [\Pi^*(p) - \Gamma^*(p)] \, \mathrm{d}p > 0.$$

By (2), it follows that the corresponding demand constraints are satisfied with slack; that is, $D_{\hat{q}}(\Pi^*) > \hat{q}$ for all anticipated quantities $\hat{q} \in (\underline{q}(p'), q^*)$. However, this means that if Π^* puts any mass (strictly) above p', then again a strict improvement is feasible. Specifically, if Π^* is supported on more than one price above p', we show that a small mean-preserving contraction above p' preserves the demand constraints (by them being slack) and increases revenue (by cross regularity). If Π^* has only one supported price above p', then satisfaction of the demand constraints for quantities right above $\underline{q}(p')$ requires a mass point also at p', and we show that revenue can be increased with a small contraction that takes mass from these two points. We thus conclude that Π^* cannot have support above p', i.e., $p' =: p^*$. This contradicts the assumption that Π^* is not greedy up to p^*.

The steps above yield that any optimal price distribution Π^* coincides with the greedy function Γ^* up to its highest supported price p^*. Theorem 1 also states that this price satisfies $p^* < \overline{v}(q^*)$. The intuition is standard: lowering prices weakly above $\overline{v}(q^*)$ to slightly below this level causes a first-order gain in revenue from marginal buyers and only a second-order loss from inframarginal buyers. Finally, $p^* < \overline{v}(q^*)$ implies that Π^* must have a mass point at p^*: while the greedy function up to p^* satisfies the demand constraints up to $\underline{q}^* = \underline{q}(p^*)$, a mass point at p^* is needed to satisfy the demand constraints over $(\underline{q}^*, q^*)$.

The proof of Theorem 1 in the Appendix proceeds via perturbations as we did here: taking a price distribution Π^* that is not greedy up to the top of its support, and showing how it can be improved while preserving the demand constraints. However, we do not build on a fixed greedy function Γ^*, nor do we rely on $\Pi^* - \Gamma^*$ being well-behaved. Instead, we show that given Π^*, we can locate an interval of anticipated quantities where the demand constraints are slack, and where we can apply contraction arguments analogous to those used in the second scenario above. For locating such an interval, concave externalities (Assumption 2) is important. To argue that contractions improve revenue while satisfying the demand constraints, a difficulty is that these constraints do not take the form of majorization as in (2) outside the linear demand environment. While this means that we cannot use off-the-shelf comparative statics on mean-preserving contractions as we did above, we show that similar comparative statics can be derived for our general model. This step extends results from Rappoport (2025); see Lemma 6 in the Appendix.

4. Benchmark comparisons

Our seller's problem features unobservable buyer types, externalities in consumption, and a concern for the strategic uncertainty that these externalities generate. In this section, we compare the seller's problem to different benchmarks, each of which removes one of these ingredients. We study how the seller's optimal policy and the induced total quantity of trade change across these benchmarks, as well as the implications for buyers' welfare. For the latter, we denote the consumer surplus associated with anticipated quantity $q \in [0, 1]$ and price $p \in \mathbb{R}_+$ by

$$\mathrm{CS}_q(p) := \int_p^{\overline{v}(q)} D_q(v)\ \mathrm{d}v,$$

and let $\mathrm{CS}_q(\Pi) := \int \mathrm{CS}_q(p)\ \mathrm{d}\Pi(p)$ for any price distribution $\Pi \in \Delta(\mathbb{R}_+)$.

4.1. Complete information

Consider a complete-information benchmark in which the seller can condition prices on both a buyer's identity $i \in I$ and his type $\theta \in \Theta$. This benchmark connects our analysis to the literature on contracting with externalities (reviewed in the Introduction). Relative to such work, one remaining difference is that our model features a continuum of buyers. While this does not affect the qualitative insights that stem from discrete-agent models, it does require adjustments to the formal treatment of the problem. We provide a complete formal analysis in the Supplementary Appendix.

Under complete information, the seller knows exactly how much each buyer (i, θ) is willing to pay for each anticipated total quantity of trade. It then follows from buyers' positive values and the presence of increasing externalities that she optimally induces all buyers to purchase. The seller's problem is to choose revenue-maximizing price offers such that all buyers purchasing is the unique equilibrium, and thus the unique rationalizable outcome (see Remark 2). In a discrete-buyer model, this boils down to choosing a so-called "divide-and-conquer" strategy (see Segal, 2003): a permutation of buyer types, along with prices that make each buyer in the permutation indifferent over purchasing if all buyers preceding him purchase and the rest do not.[23] With a continuum of buyers, we observe that the divide-and-conquer logic takes the form of a coupling between buyer types and demand quantiles. Rather than selecting a permutation of buyer types, the seller chooses a joint distribution over types and anticipated quantities, whose marginals are the type distribution and the uniform distribution on $[0, 1]$. This coupling determines, for each type, the anticipated quantity at which the buyer's price must make him indifferent between purchasing and not purchasing.

Just as the seller's choice amounts to a price distribution in our main model, here, under complete information, it amounts to a contingent price distribution, which is a measurable function $\mathbf{\Pi} : \Theta \to \Delta(\mathbb{R}_+)$. Let $\mathbf{\Pi}(\cdot|\theta)$ denote the conditional price distribution (and its CDF) for each type $\theta \in \Theta$, and let Π denote the induced marginal price distribution, so that $\Pi(p) = \int_\Theta \mathbf{\Pi}(p|\cdot)\ \mathrm{d}G$ for every price $p \geq 0$. The divide-and-conquer logic then implies that any optimal $\mathbf{\Pi}$ is greedy.[24,25]

[23] Recall that we have assumed that buyers purchase under indifference.

[24] Any optimal $\mathbf{\Pi}$ induces a distribution over $\Theta \times [0, 1]$ whose second marginal is uniform, so the induced demand satisfies $D_q(\mathbf{\Pi}) = q$ for all $q \in [0, 1]$. Hence, any optimal $\mathbf{\Pi}$ is greedy, and thus, as stated in Proposition 2, greedy up to $\overline{v}(1) \geq p^C$. The latter inequality can be strict if $u(\theta, q)$ is not supermodular.

[25] Analogous to the incomplete-information setting, perturbing the greedy contingent price

Proposition 2. *Suppose* $(\mathbf{\Pi}^C, q^C)$ *is optimal under complete information, and let* p^C *be the highest price in the support of the induced marginal* Π^C*. Then* $q^C = 1$*, and* $\mathbf{\Pi}^C$ *is greedy up to* $\overline{v}(1) \geq p^C$*. Moreover, under linear demand, each type* $\theta \in \Theta$ *is offered a price of* $\theta\overline{v}(\theta)$*, with* $\overline{v}(1) = p^C > p^*$ *and, if* $\overline{v}'(0) > 0$*, with* $\Pi^C(p) > \Pi^*(p)$ *for sufficiently small* $p > 0$*. Finally, consumer surplus* $CS_{q^C}(\mathbf{\Pi}^C)$ *can be larger or smaller than* $CS_{q^*}(\Pi^*)$*, depending on* $\overline{v}$*.*

The complete-information benchmark yields results that are qualitatively different from those described in Theorem 1 under incomplete information. As noted, under complete information the seller induces the entire population of buyers to purchase, without exclusion. Moreover, under weak conditions, no two buyers receive the same price offer, so there is no posted price. Specializing to the linear demand environment, Proposition 2 shows that the seller uses a positively assortative policy: she offers lower prices to lower types, which allows her to extract more revenue from higher types via higher prices.[26] In fact, compared to our seller's solution (Π^*, q^*) under incomplete information, the induced marginal Π^C places more weight on low prices,[27] and supports prices strictly higher than the maximum posted price p^*.

Another interesting comparison in Proposition 2 concerns consumer surplus. Because the seller wants to insulate against low-demand equilibria, she generates strictly positive consumer surplus even when buyers' types are perfectly observable. Furthermore, since she also induces a higher total quantity of trade under complete information, the resulting consumer surplus can be larger (or smaller) than under incomplete information. This speaks to the broader discussion on whether sellers' increased access to consumer information can benefit consumer welfare (e.g., Farboodi, Haghpanah and Shourideh, 2025, and references therein), and highlights a novel channel through which such gains can arise even in the extreme case of complete information.

distribution by adding a small mass at 0 makes everybody buying a unique equilibrium.

[26] This holds more generally whenever $u(\theta, q)$ is supermodular. In a discrete-buyer model, positive assortativity corresponds to a permutation that orders buyers by increasing type.

[27] The condition that $\overline{v}'(0) > 0$ is not necessary for this price ranking. For instance, our proof delivers this result for any $\overline{v}$ that is analytic at 0.

The qualitative differences between the complete-information benchmark and our incomplete-information model stem from methodological differences. Divide-and-conquer strategies like those described above are simply not available to our uninformed seller: she cannot choose a matching between buyer types and demand quantiles (or, in a discrete-buyer setting, a permutation of buyer types) when types are unobservable. Our analysis instead develops a new methodology that achieves coordination not by working through buyer types, but through anticipated quantities of trade. The seller iteratively rules out anticipated quantities as candidates for equilibria, but she cannot control which buyer types are induced to purchase at each of these quantities. As a consequence, the incomplete-information solution generates non-assortative price dispersion and exclusion patterns—as stated in Corollary 2, and in contrast with the complete-information solution.

4.2. No externalities

Consider next a benchmark without externalities in consumption. Specifically, suppose buyers' willingness to pay is independent of the anticipated total quantity of trade. If a buyer of type $\theta \in \Theta$ purchases at price p_i, then his payoff is $u(\theta, \overline{q}) - p_i$ regardless of the equilibrium total quantity q, where $\overline{q} \in (0, 1]$ is an exogenous demand parameter. This benchmark corresponds to the canonical monopoly problem with incomplete information. Using the results of Myerson (1981), it then follows that the seller's optimal pricing policy is a posted price. No equilibrium multiplicity arises under no externalities, and by Myerson regularity—which is implied by cross regularity (Assumption 3)—the optimal posted price is the unique $p^N \in (0, \overline{v}(\overline{q}))$ with $\varphi_{\overline{q},\overline{q}}(p^N) = 0$.

Proposition 3. *Suppose (Π^N, q^N) is optimal under no externalities with demand parameter $\overline{q} \in (0, 1]$, and let p^N be the highest price in the support of Π^N. Then $q^N < 1$, and Π^N is degenerate on $p^N < \overline{v}(\overline{q})$. Moreover, under linear demand, p^N induces total quantity $q^N = 1/2 < q^*$, whereas the ranking of p^N and p^*, and that of consumer surplus $CS_{\overline{q}}(p^N)$ and $CS_{q^*}(\Pi^*)$, depend on $(\overline{v}, \overline{q})$. For $\overline{q} = 1$, there exists $\overline{v}$ such that $p^N > p^*$ and $CS_{\overline{q}}(p^N) < CS_{q^*}(\Pi^*)$.*

Relative to our main model with externalities (and worst-case selection), an immediate difference in the no-externality benchmark is the absence of price dispersion, as all buyers face the same price p^N. As a result, exclusion is assortative—only buyers with value $u(\theta, \overline{q}) < p^N$ choose not to buy—again in contrast with our main model.

In the linear demand environment, the optimal no-externality price is $p^N = \overline{v}(\overline{q})/2$, which yields total quantity $q^N = D_{\overline{q}}(p^N) = 1/2$. This price can be higher or lower than the highest supported price p^* in our seller's solution (Π^*, q^*), depending on the function $\overline{v}$ and the demand parameter $\overline{q}$. Yet, we show that $p^* < \overline{v}(q^*)/2$, implying that the induced total quantity under no externalities is strictly smaller than in our seller's solution, since $1/2 < D_{q^*}(p^*) \leq D_{q^*}(\Pi^*) = q^*$. Intuitively, the seller gains less from increasing demand when buyers' purchasing decisions are no longer complementary.

Proposition 3 also reports that the ranking of consumer surplus is ambiguous. This reflects not only the ambiguity in the ranking of prices p^N and p^*, but also the fact that under no externalities, buyers' welfare depends directly on the demand parameter $\overline{q}$. If we take $\overline{q} = 1$, then all buyers have higher values in the no-externality benchmark than in our main model with externalities. Despite this, the proposition shows that consumer surplus can be strictly higher in our main model, as a consequence of lower prices and a higher induced total quantity under our seller's policy. This is indeed the case in the example discussed in the Introduction, depicted in the top panel of Figure 2.

4.3. Best-case selection

The last benchmark we study retains both incomplete information and externalities, but departs from our main model in that the seller has no concern for strategic uncertainty. Specifically, suppose that for any price distribution $\Pi \in \Delta(\mathbb{R}_+)$ that the seller chooses, she can select the equilibrium that buyers play in the induced game if multiple equilibria arise. Rather than being concerned with the worst case as in program (P), such a seller maximizes revenue in the best-case equilibrium:

$$\sup_{\Pi \in \Delta(\mathbb{R}_+)} \max_{q \in [0,1]} R_q(\Pi)$$
$$\text{subject to} \quad D_q(\Pi) = q.$$

Proposition 4 below shows that the solution to this program takes the form of a posted price, as in the no-externality benchmark studied in the previous section. Thus, so long as the seller can select her preferred equilibrium, the presence of externalities does not alter the nature of the optimal mechanism.[28] The argument builds on Bulow and Roberts (1989). Recall that by Myerson regularity—which is implied by cross regularity (Assumption 3)—the seller's revenue from a given price is a concave function of the quantity demanded at that price. Fix any nondegenerate price distribution Π and suppose it induces a best-case equilibrium total quantity $q > 0$. This total quantity is generated by aggregating purchases at different prices across buyers. Concavity then implies that concentrating these sales at a single price increases revenue, provided that the total quantity demanded is kept unchanged. Accordingly, the seller can improve upon Π with a q-preserving posted price, that is, by offering each buyer a price $p = D_q^{-1}(q) \in [0, \overline{v}(q)]$.

Proposition 4. *Suppose (Π^B, q^B) is optimal under best-case selection, and let p^B be the highest price in the support of Π^B. Then $q^B < 1$, and Π^B is degenerate on $p^B < \overline{v}(q^B)$. Moreover, under linear demand, this price satisfies $p^B < p^*$, induces total quantity $q^B < q^*$, and yields consumer surplus $CS_{q^B}(p^B) < CS_{q^*}(\Pi^*)$.*

Specializing to the linear demand environment, we find that the optimal best-case price p^B is strictly lower than the highest supported price p^* in

[28] The externalities do affect the level of the optimal posted price, as they make the quantity demanded more responsive to price changes. Thus, defining p^B and q^B as in Proposition 4 below, and taking $\overline{q} = q^B$ in the no-externality setting, we obtain $p^B \leq p^N$, with strict inequality under sufficient smoothness. See Section 17.2 of Easley and Kleinberg (2010) for a related discussion.

our seller's solution (Π^*, q^*). Thus, not all buyers benefit when the seller is concerned with worst-case outcomes: some buyers receive generous discounts as the seller seeks to ensure a high demand, while others receive price offers strictly higher than p^B. At the same time, Proposition 4 tells us that the total quantity demanded is higher than in the best-case benchmark, which means that buyers on average do purchase at a lower price in the worst-case solution. Interestingly, while the seller is concerned with ruling out low-quantity equilibria—and must therefore offer discounts and sacrifice revenue to ensure any given total quantity—she ends up inducing a higher total quantity of trade than in the absence of this concern. Proposition 4 further establishes that, as a consequence, consumer surplus increases due to the seller's worst-case focus.

We next sketch the argument behind the price and quantity comparisons in Proposition 4. By Theorem 1, any optimal price distribution in our main model is a function $\Pi(\cdot|\hat{p})$ that coincides with the greedy function (unique under linear demand) up to a highest supported price $\hat{p}$. Define $\mathcal{R}(\hat{p}, \hat{q})$ as the seller's worst-case revenue given $\Pi(\cdot|\hat{p})$ and a buyers' anticipated quantity $\hat{q}$. In a worst-case equilibrium, $\hat{q}$ is equal to the lowest quantity demanded given that $\Pi(\cdot|\hat{p})$ is the limit worst-case price distribution; call it $\mathcal{Q}(\hat{p})$. Then $\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))$ gives the seller's worst-case revenue parametrized by $\hat{p}$. We define analogous objects for the best-case problem, with $\mathcal{R}^B(\hat{p}, \hat{q})$ being the seller's best-case revenue given a posted price $\hat{p}$ and anticipated quantity $\hat{q}$, and $\mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p}))$ taking $\hat{q}$ to equal the highest equilibrium quantity $\mathcal{Q}^B(\hat{p})$ under $\hat{p}$.

Our analysis is facilitated by the fact that, in the linear demand environment, these worst-case and best-case revenue functions are strictly quasiconcave, with unique interior optima p^* and p^B given by[29]

$$\left.\frac{\mathrm{d}\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))}{\mathrm{d}\hat{p}}\right|_{\hat{p}=p^*} = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}\mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p}))}{\mathrm{d}\hat{p}}\right|_{\hat{p}=p^B} = 0. \tag{6}$$

[29] The fact that the worst-case problem is strictly quasiconcave under linear demand also plays a role in our proofs of Proposition 2 and Proposition 3.

Hence, to establish the ranking between p^* and p^B, it suffices to sign

$$\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\hat{p}} = \underbrace{\frac{\partial \mathcal{R}}{\partial \hat{p}}}_{\substack{\text{monopoly} \\ \text{effect}}} + \underbrace{\frac{\partial \mathcal{R}}{\partial \hat{q}} \frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}\hat{p}}}_{\substack{\text{externality} \\ \text{effect}}} \tag{7}$$

at $\hat{p} = p^B$. The first term on the right-hand side is the **monopoly effect**. This effect tells us how revenue changes with $\hat{p}$ while keeping the anticipated quantity $\hat{q}$, and thus the demand function $D_{\hat{q}}$, fixed. The second term is the **externality effect**. This effect tells us how revenue changes as the demand function $D_{\hat{q}}$ shifts towards the new equilibrium—that is, given that the anticipated quantity $\hat{q}$ must adjust to match the quantity demanded $\mathcal{Q}(\hat{p})$ following an increase in $\hat{p}$. We show that conditional on pricing at $\hat{p}$ (and scaling by an appropriate normalizing factor), the worst-case monopoly effect of raising $\hat{p}$ starting from $\hat{p} = p^B$ is higher (i.e., more positive) than the analogous best-case monopoly effect, and the worst-case externality effect is also higher (i.e., less negative) than the analogous best-case externality effect. Hence, given the definition of p^B in (6), the monopoly and externality effects imply $\mathrm{d}\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))/\mathrm{d}\hat{p} > 0$ at $\hat{p} = p^B$. We conclude that the worst-case highest price p^* is strictly higher than the best-case posted price p^B.

The idea behind the ranking of the worst-case and best-case quantities, q^* and q^B, is similar. We show that $\mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p}))$ increases as $\hat{p}$ is reduced from a level that makes the worst-case equilibrium quantity equal to q^B, and therefore the optimal such quantity must satisfy $q^* > q^B$. It is worth noting that this finding contrasts with results in complete-information settings. As Segal (2003) shows, under complete information and general conditions, a robustness requirement in the form of worst-case or unique implementation would only lower the total quantity of trade relative to best-case implementation.

5. The strength of externalities

Externalities in buyers' purchasing decisions are a key novel ingredient of our seller's problem. In Section 5.1, we examine how the seller's solution changes as these externalities become stronger. In Section 5.2, we consider a setting in which buyers belong to groups with heterogeneous strength of externalities, and the seller's price offers can condition on buyer group. Throughout this section, we focus on the linear demand environment.

5.1. Comparative statics

Recall that under linear demand, a buyer's willingness to pay given type $\theta \in \Theta$ and anticipated quantity $q \in [0,1]$ is $u(\theta, q) = \theta\overline{v}(q)$. The strength of externalities in consumption is reflected in the function $\overline{v}$.

Definition 2. *Under linear demand, say $\overline{v}_1$ has* **stronger externalities** *than $\overline{v}_0$ if, for all $q \in (0,1]$,*

(i) $\overline{v}_1(q) > \overline{v}_0(q)$, and

(ii) $\overline{v}_1(q)/\overline{v}_0(q)$ is strictly increasing in q.

Intuitively, buyers' purchasing decisions are more complementary if their willingness to pay grows with the anticipated total quantity demanded at a higher rate, as captured by condition (ii).[30] Condition (i) additionally says that buyers' willingness to pay at any given anticipated quantity is higher when externalities are stronger; since our model assumes $\overline{v}(0) = 0$, this would follow from (ii) if $\overline{v}_1'(0) > \overline{v}_0'(0)$.

Using Definition 2, we find the following comparative statics.

Proposition 5. *Take the linear demand environment. Suppose $\overline{v}_1$ has stronger externalities than $\overline{v}_0$, with corresponding optimal price distributions Π_1^* and Π_0^*. Relative to Π_0^*, then Π_1^* induces a higher total quantity $q_1^* > q_0^*$. Moreover, Π_1^* puts lower weight on low prices: $\Pi_1^*(p) < \Pi_0^*(p)$ for all strictly positive $p < \min\{p_1^*, p_0^*\}$.*

Stronger externalities make it less costly for the seller to insulate against low-demand equilibria. Specifically, take any anticipated quantity $\hat{q} \in (0,1)$

[30] Observe that taking $\overline{v}_1'(\cdot) > \overline{v}_0'(\cdot)$ would not yield the desired order: multiplying $\overline{v}$ by a constant $\kappa > 0$ has no effect on the seller's solution up to a change of numéraire.

and greedy prices over $[0, \overline{v}(\hat{q}))$ that satisfy the demand constraints in program (P*) up to $\hat{q}$. Under stronger externalities, because buyers' willingness to pay is more responsive to anticipated quantity, the seller can satisfy the demand constraint at $\hat{q}$ without the need to offer prices below $\overline{v}(\hat{q})$ to such a large mass of buyers. As a result, the optimal price distribution places relatively less weight on discounted prices below a given posted price. Moreover, because the seller can guarantee a given equilibrium quantity while charging higher prices, she optimally induces a higher total quantity when externalities are stronger. Together, these price and quantity effects explain why Proposition 5 does not pin down how the posted price itself changes with the strength of externalities.

For illustration, we can compare the linear-demand examples shown in the top and middle panels of Figure 2. The second example (with $\overline{v}(q) = q+q^2$) has stronger externalities than the first example (with $\overline{v}(q) = q$), and accordingly induces a higher total quantity of trade (as reported in the figure caption). The second example also exhibits a higher posted price and lower weight on discounted prices below the first-example posted price.

The proof of the comparative static concerning the seller's optimal price distribution follows directly from equation (3), which defines the unique greedy function under linear demand. If $\overline{v}_1$ has stronger externalities than $\overline{v}_0$, then the greedy function under $\overline{v}_1$ is lower than that under $\overline{v}_0$ in the first-order stochastic dominance (FOSD) sense.

To prove the comparative static for the optimal total quantity, we use arguments similar to those described in Section 4.3. Given $\overline{v}_1$ and $\overline{v}_0$, let $\mathcal{R}_1(\hat{p}, \mathcal{Q}_1(\hat{p}))$ and $\mathcal{R}_0(\hat{p}, \mathcal{Q}_0(\hat{p}))$ be the respective revenue functions parametrized by the highest offered price $\hat{p}$. We study how the strong-externality revenue $\mathcal{R}_1$ changes as we increase the highest price $\hat{p}$, starting from a level that makes the induced strong-externality quantity equal to the optimal weak-externality quantity q_0^*.[31] By the FOSD ranking of the greedy functions, such a starting level for $\hat{p}$ is strictly higher than p_0^*. We then show that increasing $\hat{p}$ from

[31] In the formal proof, we consider the case of $\overline{v}_1$ having *marginally* stronger externalities, and use a continuous deformation argument for the general case.

that level causes strong-externality monopoly and externality effects, as defined in equation (7), which are both lower (i.e., less positive or more negative) than the corresponding weak-externality effects caused by increasing $\hat{p}$ from p_0^*. Since the latter weak-externality effects add to zero by definition of p_0^*, this means that the strong-externality revenue $\mathcal{R}_1$ can be increased by lowering $\hat{p}$. Thus, we obtain that the optimal total quantities satisfy $q_1^* > q_0^*$, as stated in Proposition 5.

5.2. Heterogeneity

Different groups of buyers may exhibit different strength of externalities. For example, consider a seller of file-sharing services. Because these services are more heavily used in the corporate sector, corporate buyers' willingness to pay tends to be higher and to grow at a faster rate with total adoption than that of retail buyers. How should the seller's price offers take this heterogeneity into account?

We consider $N > 1$ buyer groups indexed by $n \in \{1, \dots, N\}$, each making up a proportion $\lambda_n > 0$ of the population, with $\sum_n \lambda_n = 1$. A buyer's willingness to pay is increasing in the anticipated quantity q demanded by all buyers, but this externality is stronger for buyers in higher-indexed groups. Specifically, in the linear demand environment with $u(\theta, q) = \theta\overline{v}(q)$, and consistent with Definition 2, we assume that for all $q \in [0, 1]$ and all $n \in \{1, \dots, N-1\}$: (*i*) $\overline{v}_{n+1}(q) > \overline{v}_n(q)$, and (*ii*) $\overline{v}_{n+1}(q)/\overline{v}_n(q)$ is strictly increasing in q.

The seller's price offers can condition on both a buyer's identity i and the group n to which he belongs (but not on the buyer's private type θ). The seller's problem thus amounts to choosing a price distribution $\Pi_n \in \Delta(\mathbb{R}_+)$ for each group $n \in \{1, \dots, N\}$, with the objective of maximizing her total worst-case revenue. Given an anticipated total quantity $q \in [0, 1]$ and a price $p \in \mathbb{R}_+$, denote the (unweighted) quantity demanded by group-n buyers by $D_{n,q}(p) := 1 - p/\overline{v}_n(q)$, and let $D_{n,q}(\Pi_n) := \int D_{n,q}(p)\, \mathrm{d}\Pi_n(p)$ and $R_{n,q}(\Pi_n) := \int pD_{n,q}(p)\, \mathrm{d}\Pi_n(p)$ for any $\Pi_n \in \Delta(\mathbb{R}_+)$. Applying the logic of Proposition 1, we can write the seller's problem analogously to (P*) in the main model:

$$\max_{\{\Pi_n \in \Delta(\mathbb{R}_+)\}_n, q \in [0,1]} \sum_n \lambda_n R_{n,q}(\Pi_n) \tag{P_N^*}$$
$$\text{subject to} \quad \sum_n \lambda_n D_{n,\hat{q}}(\Pi_n) \geq \hat{q} \quad \forall \hat{q} \in (0, q).$$

Implementing an equilibrium quantity $q \in [0,1]$ requires ruling out all lower quantities as equilibria. Thus, as in (P*), the demand constraints in (P_N^*) say that the total quantity demanded at each anticipated quantity $\hat{q} \in (0, q)$ must exceed $\hat{q}$.[32] Here, the total quantity demanded is the sum of the demands from each of the N buyer groups. The seller therefore chooses how to use the different groups to build demand up to q. We next define a class of price distributions that build demand in an ordered manner. Let $\underline{q}_n := \overline{v}_n^{-1}$.

Definition 3. *Given prices $p_1 \leq \cdots \leq p_N$, say price distributions $(\Pi_n)_{n=1}^N$ are **residual greedy up to $(p_n)_{n=1}^N$** if the following holds for all $n \in \{1, \ldots, N\}$:*

- *if $p_n \leq \overline{v}_n(q_{n-1})$, then Π_n is degenerate on p_n,*
- *if $p_n > \overline{v}_n(q_{n-1})$, then $\text{supp}(\Pi_n) = [\overline{v}_n(q_{n-1}), p_n]$ and*

$$\sum_{m=1}^{n} \lambda_m D_{m,\hat{q}}(\Pi_m) = \hat{q} \quad \forall \hat{q} \in (q_{n-1}, \underline{q}_n(p_n)),$$

where $q_n := \max\{q \in [0,1] : \sum_{m=1}^n \lambda_m D_{m,q}(\Pi_m) = q\}$.

Price distributions $(\Pi_n)_n$ that are residual greedy up to $(p_n)_n$ have two key properties. First, since the quantities $(q_n)_n$ as defined satisfy $p_n \leq \overline{v}_n(q_n)$ and $q_1 < \ldots < q_N$, the supports of these distributions are ordered.[33] This means that all group-n buyers face lower prices than any buyer in group $n+1$, and only buyers from groups $\{1, \ldots, n\}$ are used to satisfy the demand

[32] As in (P*), we impose the demand constraints as weak inequalities, with the solution being the limit of a sequence $((\Pi_{n,k})_n, q_k)_k$ that satisfies them strictly for every k.

[33] Observe that there is no circularity in Definition 3 since Π_n depends on $(q_1, \ldots, q_{n-1})$ while q_n depends on Π_n.

constraints up to anticipated quantity q_n. Second, given the price distributions for groups $\{1, \ldots, n-1\}$, the group-n price distribution makes the demand constraints for $\hat{q} \in (q_{n-1}, \underline{q}_n(p_n))$ hold with equality. Intuitively, the seller follows a greedy procedure as in our main model, but since the constraints aggregate the quantity demanded by buyers in all groups $(1, \ldots, n)$, the prices are greedy in a residual sense: group-n buyers are offered discounts only as needed to build the residual demand not fulfilled by lower-index-group buyers.

We show that the seller's optimal pricing policy consists of price distributions that are residual greedy up to the top of their supports.

Proposition 6. *Take the linear demand environment with buyer groups $n \in \{1, \ldots, N\}$ indexed by increasing strength of externalities. Suppose $((\Pi_n^*)_{n=1}^N, q^*)$ is optimal, and let p_n^* be the highest price in the support of Π_n^*. Then the price distributions $(\Pi_n^*)_{n=1}^N$ are residual greedy up to $(p_n^*)_{n=1}^N$, and Π_N^* has a mass point at $p_N^* < \overline{v}_N(q^*)$. Therefore, for each $n \in \{1, \ldots, N-1\}$,*

$$\max \operatorname{supp}(\Pi_n^*) \leq \min \operatorname{supp}(\Pi_{n+1}^*).$$

Moreover, this inequality is strict unless $p_n^* = p_{n+1}^* = 0$.

This result sheds light on how the seller optimally builds demand toward an equilibrium total quantity. Buyers from strong-externality groups are more responsive to the anticipated total quantity of trade than those from weak-externality groups. Hence, the seller offers lower prices to weak-externality buyers to provide assurance of a higher total quantity to strong-externality buyers, thereby allowing her to extract higher revenue from the latter.[34] Going back to the example of a seller of file sharing services, Proposition 6 says that all retail buyers will enjoy larger discounts than any corporate buyer.

The proposition further shows that the methodology from our main model extends to this setting with heterogeneous buyer groups. Once we establish that the price distributions $(\Pi_n^*)_n$ have ordered supports—more specifically, that any prices $(p_n)_n$ respectively in the supports of $(\Pi_n^*)_n$ have $\underline{q}_n(p_n) \leq$

[34] Related results appear in Chan (2021) and Nora and Winter (2024).

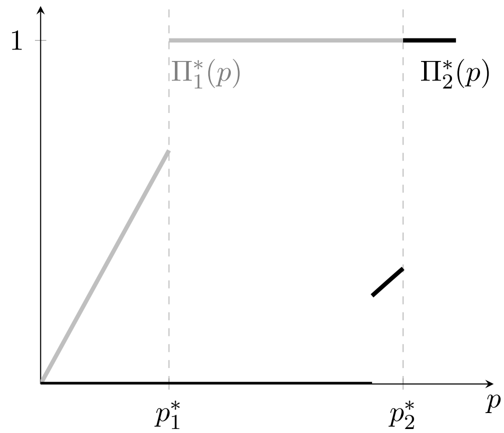

Figure 4: Optimal price distributions for a population with 2 buyer groups of equal weight, where group-1 and group-2 buyers have willingness to pay as in the first and second examples of Figure 2 respectively. For group 1 (light gray line), we obtain $\Pi_1^*(p) = 4p$ for $p < p_1^*$ and $\Pi_1^*(p) = 1$ for $p \geq p_1^*$, with $p_1^* \approx 0.17$ and $q_1 \approx 0.33$. For group 2 (black line), we obtain $\Pi_2^*(p) = 0$ for $p < \overline{v}_2(q_1)$, $\Pi_2^*(p) = (1 + 6p - 2\sqrt{1+4p} + p_1^*(1 - 2p_1^*))/\sqrt{1+4p}$ for $\overline{v}_2(q_1) \leq p < p_2^*$, and $\Pi_2^*(p) = 1$ for $p \geq p_2^*$, where $p_2^* \approx 0.48$ and $q_2 = q^* \approx 0.74$.

$\underline{q}_{n+1}(p_{n+1})$—we can then apply the arguments of Theorem 1 to each of the N groups. This yields the characterization in Proposition 6, with price distributions that are residual greedy up to the top of their supports, and a mass point at p_N^*, as well as possibly mass at other points in $\{p_1^*, \ldots, p_{N-1}^*\}$.[35] The interpretation is intuitive: the seller sets a posted price, along with group-exclusive discounts and personalized discounts within each group.

Figure 4 provides an illustration. We consider a population with $N = 2$ buyer groups of equal weight. Group-1 and group-2 buyers have willingness to pay as given, respectively, in the first and second examples of Figure 2. We can interpret the seller's solution as posting the price p_2^* and offering all buyers in the weak-externality group 1 a group-exclusive discount of $p_2^* - p_1^*$, in addition to personalized discounts to some buyers in this group and to some buyers in the strong-externality group 2. In this way, the seller builds demand

[35] Observe that residual greediness can be consistent with a price distribution Π_n^* also having a mass point at the lower limit of its support. An example is shown in Figure 4.

with group-1 buyers up to a quantity q_1, and then extracts higher revenue from group-2 buyers as she continues to grow demand with buyers from both groups up to the equilibrium quantity $q^* > q_1$.

6. Discussion

In this section, we discuss several variants of our model and examine how our analysis would (or would not) change.

Screening menus. We have phrased our model with the seller choosing personalized price offers. Since buyers have private information about their payoff types, it is natural to ask whether the seller could do better with more sophisticated mechanisms. We argue here that our focus on price offers is without loss of generality within the class of public bilateral contracts.

Let $\mathcal{M}$ denote the set of all compact subsets of $[0,1]\times\mathbb{R}_+$ that contain $(0,0)$. We consider a general contracting environment in which the seller offers a menu $M_i \in \mathcal{M}$ to each buyer $(i,\theta) \in I\times\Theta$, and buyers then simultaneously choose an option from their respective menus. Each menu option specifies a probability of trade $x \in [0,1]$ and a transfer $t \in \mathbb{R}_+$ from the buyer to the seller, with $(0,0)$ corresponding to a buyer's option of not purchasing the good and not making any transfer. Clearly, this is a generalization of our main model, as menus in $\mathcal{M}^P := \{\{(0,0),(1,p)\} : \ p \in \mathbb{R}_+\}$ correspond exactly to price offers.

For any menu $M \in \mathcal{M}$ and willingness to pay $v \in \mathbb{R}_+$, let $(x_M(v), t_M(v))$ be the element of $\arg\max_{(x,t)\in M}(xv-t)$ with highest x. If a buyer anticipates total quantity of trade q and faces menu offer M, the expectation of his quantity demanded is $D_q(M) := \int_0^{\overline{v}(q)} x_M(v) f_q(v)\ \mathrm{d}v$, and the expected revenue he generates is $R_q(M) := \int_0^{\overline{v}(q)} t_M(v) f_q(v)\ \mathrm{d}v$.[36] Analogous to our main model, we can summarize the seller's mechanism choice via a distribution—here, a distribution $\mu \in \Delta\mathcal{M}$ over menu offers. Given such a μ, a total quantity q is an equilibrium quantity if and only if $q = D_q(\mu) := \int D_q(M)\ \mathrm{d}\mu(M)$, and the resulting revenue is $R_q(\mu) := \int R_q(M)\ \mathrm{d}\mu(M)$.

[36] The assumption that buyers choose the highest-x option among their preferred menu options corresponds to our main model's assumption that buyers purchase when indifferent.

We argue that any menu distribution $\mu \in \Delta\mathcal{M}$ admits some price distribution $\Pi_\mu \in \Delta(\mathbb{R}_+)$ with the same set of equilibrium quantities $q \in [0,1]$ and generating the same revenue for every equilibrium quantity. The idea is simple. First, it follows by standard arguments (Myerson, 1981) that any menu $M \in \mathcal{M}$ can be replaced by a revenue-equivalent random posted price. That is, given M, we can define a distribution Π_M such that a buyer who has willingness to pay $v \in [0, \overline{v}(1)]$ and faces a random posted price with distribution Π_M (and purchases whenever doing so is weakly optimal) would then purchase with probability $x_M(v)$ and generate an expected transfer of $t_M(v)$.[37] Thus, for any $q \in [0,1]$, we obtain $D_q(\Pi_M) = D_q(M)$ and $R_q(\Pi_M) = R_q(M)$. Next, because there is a continuum of buyers, we can take the distribution of prices that the individual random posted prices generate in the population and implement it directly as a distribution of price offers. That is, we can define Π_μ to be the barycenter $\int \Pi_M \, \mathrm{d}\mu(M)$, yielding $D_q(\Pi_\mu) = D_q(\mu)$ and $R_q(\Pi_\mu) = R_q(\mu)$ for every $q \in [0,1]$.

The implication is that our focus on price offers rather than menu offers is without loss. Instead, what matters for our analysis is our maintained assumption that contracts are bilateral and public. Bilateral contracts means that the offer to a buyer cannot directly condition on the purchasing decisions of other buyers. If such multilateral contract offers were feasible, they could mitigate the seller's concern for strategic uncertainty.[38] Multilateral contracts are often difficult to verify and enforce in practice, and for this reason they are commonly ruled out in the contracting-with-externalities literature.[39] Finally, public contracts means that buyers know the realized distribution of prices.[40] Whether revenue can be improved in a setting with private contracts—as is the case in the moral-hazard problem of Halac et al. (2021)—is an open question.

[37] Taking $\nu > 0$, we can let $\Pi_M(p) = x_M(p)$ for $p < \overline{v}(1) + \nu$, and $\Pi_M(p) = 1$ otherwise.

[38] For instance, the seller would be able to guarantee the best-case pricing outcome by offering each buyer a contract that specifies the best-case price p^B conditional on total quantity $q \geq q^B$, and a zero price otherwise.

[39] See, e.g., Innes and Sexton (1994), Segal (2003), and Halac et al. (2020).

[40] This assumption matters for showing the optimality of our seller's policy but not necessarily for its implementation. See our concluding remarks in Section 7.

Warm start. Our model assumes a cold-start problem: no buyer is willing to purchase at a strictly positive price if he anticipates that no other buyer will purchase. Formally, we assumed that the highest willingness to pay as a function of the anticipated total quantity of trade, $\overline{v}(q)$, satisfies $\overline{v}(0) = 0$. We now discuss how our results change if we relax this assumption and consider instead a "warm-start" model with $\overline{v}(0) > 0$. We maintain all of our other assumptions, including that $\overline{v}$ is continuously differentiable with $\overline{v}' > 0$.

Conceptually, our analysis extends to the warm-start model with little modification. Both our restatement of the seller's problem in Proposition 1 and our characterization of the seller's solution in Theorem 1 continue to apply. In particular, any optimal (Π^*, q^*) has Π^* greedy up to its highest supported price p^*, so our notion of greediness remains useful for describing the seller's optimal price distribution. A key difference, however, is that greediness now implies zero mass on prices strictly below $\overline{v}(0)$, so we must have $\Pi^*(p) = 0$ for all $p < \min\{p^*, \overline{v}(0)\}$. It follows that in the warm-start model, Π^* takes one of two forms: either it has no supported prices strictly below $\overline{v}(0)$—in which case it is a posted price with dispersed discounts, as in our cold-start model—or it is degenerate on some $p^* < \overline{v}(0)$, in which case it is simply a posted price.

The intuition for why a degenerate price distribution can be optimal here is immediate when $\overline{v}(0)$ is sufficiently large. Denote by $p^N(\overline{q})$ the optimal no-externality posted price under exogenous demand parameter $\overline{q}$. Our analysis implies that any optimal (Π^*, q^*) has highest supported price $p^* \leq p^N(q^*)$. Hence, a sufficient condition for the seller to choose a degenerate price distribution in the warm-start model is $\overline{v}(0) > p^N(1)$. In this case, the externalities in consumption operate in a region of highest values that lies above the highest price the seller would ever want to offer. The seller therefore cannot gain from setting prices above $\overline{v}(0)$, and thus cannot gain from price dispersion.

An extreme case is an environment in which the support of willingness to pay is independent of the anticipated total quantity of trade. That is, suppose that we depart from our maintained assumption that $\overline{v}$ is strictly increasing and consider a fixed-support case with $0 < \overline{v}(0) = \overline{v}(1)$. Then an optimal policy is necessarily a posted price. When the support of θ is bounded, however, this fixed-support case is inconsistent with strictly positive network externalities: variation of the support of willingness to pay with the anticipated total quantity is a direct implication of the utility function $u(\theta, q)$ being strictly increasing in θ and q. We thus view the optimality of a posted price in this case not as a limitation, but rather as reinforcing the paper's message about the role of externalities in driving price dispersion.

Low-value externalities. We have assumed that the lowest buyer value is zero for all anticipated quantities; i.e., that F_q has support $[0, \overline{v}(q)]$ for all $q \in [0,1]$. Suppose instead that the support of F_q is $[\underline{v}(q), \overline{v}(q)]$, where $\underline{v}$ is continuously differentiable with $\underline{v}' \geq 0$. Adapting our concavity assumptions to this more general setting, we can show that our main result Theorem 1 goes through essentially unchanged. The proof would combine demand-preserving contractions as the ones used in our baseline model together with some price increases below $\underline{v}(q^*)$, namely in a price range where all buyers are willing to purchase in equilibrium.

7. Concluding remarks

We have presented a framework for studying personalized pricing in markets with incomplete information and network externalities. Our analysis provides an explanation for the use of posted prices together with dispersed discounts, as well as for resulting patterns of price dispersion and exclusion that are non-assortative.

We believe there are several potentially fruitful directions for future research. For example, one could build on our model to examine the possibility of congestion in consumption; this could be introduced by assuming that buyers' highest-value function $\overline{v}(q)$ is inverse-U-shaped in the anticipated total quantity of trade q. Another interesting direction would be to extend our analysis to a two-sided platform, say with sellers on one side and buyers on the other.[41] Unlike in our heterogeneous-groups setting of Section 5.2, here

[41] See Jullien, Pavan and Rysman (2023) for a recent survey of the literature on two-sided

participants on each side would have a value of participating that is increasing in the number of participants on the other side but (weakly) decreasing in the number of participants on their same side.

Finally, a natural extension of our analysis would be to introduce dynamics. Suppose buyers receive offers from the seller and can hold onto them, so that they decide not only whether to purchase but also when to purchase. Taking others' decisions to be independent of his own, assume a buyer purchases at a time $t \geq 0$ if and only if doing so is dominant given the publicly observed quantity of purchases up to time $t - 1$.[42] If the seller offers each buyer a constant price, then her solution coincides with that in our static model. In fact, this dynamic setting provides a transparent implementation of our seller's solution in which buyers need not know either the seller's price distribution or the distribution of other buyers' types; instead, they only need to observe the total quantity purchased over time (and their own price offer). A simple example that fits this description is the sale of tickets for an event, such as a concert. Sellers often offer personalized discounts by restricting resale, and the publicly available "seat map" lets buyers observe at each point how many other buyers have purchased tickets so far. In future work, we plan to study the conditions under which our solution remains optimal in this dynamic setting even when the seller can commit to prices that change over time.

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A. Preliminaries

Lemma 1. *Continuous bounded functions $\psi_D, \psi_R : \mathbb{R}_+ \to \mathbb{R}_+$ exist such that each $Y \in \{D, R\}$, price distribution Π, and quantity $q \in [0,1]$ have*

$$Y_q(\Pi) = \int_\Theta \int_{\mathbb{R}_+} \mathbf{1}_{p \leq u(\theta,q)} \psi_Y(p) \; \mathrm{d}\Pi(p) \; \mathrm{d}G(\theta);$$

and $\psi_D(p), \psi_R(p) > 0$ for any $p > 0$.

Proof. Define $\psi_D, \psi_R : \mathbb{R}_+ \to \mathbb{R}_+$ via $\psi_D(p) := 1$ and $\psi_R(p) := \min\{\overline{v}(1),\ p\}$. Fixing $Y \in \{D, R\}$, observe that any price distribution Π and quantity $q \in [0,1]$ have

$$Y_q(\Pi) = \int_\Theta \int_{\mathbb{R}_+} \mathbf{1}_{p \leq u(\theta,q)} \psi_Y(p) \; \mathrm{d}\Pi(p) \; \mathrm{d}G(\theta).$$

In particular, to see this formula holds for $Y = R$, note that $u(\theta, q) \leq \overline{v}(q) \leq \overline{v}(1)$ for almost every θ. Evidently, both functions are continuous, bounded, and have the required strict positivity property. *Q.E.D.*

Lemma 2. *For any $\Pi \in \Delta(\mathbb{R}_+)$, the two functions $q \mapsto D_q(\Pi), R_q(\Pi)$ are increasing and right-continuous, and are strictly comonotonic with each other.*[43] *Hence, a highest and lowest equilibrium quantity exist, and these are the (unique) best-case and worst-case equilibria, respectively.*

Proof. Consider any $Y \in \{D, R\}$, and let ψ_Y be as given by Lemma 1. Because u is increasing in its second argument, any $q, \hat{q} \in [0,1]$ with $q < \hat{q}$ have

$$Y_{\hat{q}}(\Pi) - Y_q(\Pi) = \int_\Theta \int_{\mathbb{R}_+} \mathbf{1}_{u(\theta,q) < p \leq u(\theta,\hat{q})} \psi_Y(p) \; \mathrm{d}\Pi(p) \; \mathrm{d}G(\theta).$$

The integrand $(\theta, p) \mapsto \mathbf{1}_{u(\theta,q) < p \leq u(\theta,\hat{q})} \psi_Y(p)$ is globally nonnegative, telling us $q \mapsto Y_q(\Pi)$ is weakly increasing. Moreover, that $u \geq 0$ implies the integrand

[43] That is, $D_{\hat{q}}(\Pi) > D_q(\Pi)$ if and only if $R_{\hat{q}}(\Pi) > R_q(\Pi)$.

globally has the same sign as $(\theta, p) \mapsto \mathbf{1}_{u(\theta,q)<p\leq u(\theta,\hat{q})}$, so that $Y_{\hat{q}}(\Pi) > Y_q(\Pi)$ if and only if $u(\theta, q) < p \leq u(\theta, \hat{q})$ for a nonzero measure of (θ, p). Because this condition is the same for both Y, it follows that $q \mapsto D_q(\Pi), R_q(\Pi)$ are strictly comonotonic. Finally, the bounded integrand converges pointwise to zero as $\hat{q} \searrow q$ because u is continuous in its second argument, meaning right continuity follows from the Lebesgue dominated convergence theorem.

We conclude with the last assertion. A highest and lowest equilibrium quantity both exist by the Knaster-Tarski theorem, the self map $q \mapsto D_q(\Pi)$ on $[0, 1]$ being increasing. Next, observe any two equilibrium quantities $q < \hat{q}$ have $D_{\hat{q}}(\Pi) - D_q(\Pi) = \hat{q} - q > 0$, and so strict comonotonicity means $R_{\hat{q}}(\Pi) > R_q(\Pi)$. Therefore the highest [resp. lowest] equilibrium quantity is the unique best [resp. worst] one for the seller. *Q.E.D.*

Lemma 3. *Endowing $\Delta(\mathbb{R}_+)$ with the topology of weak convergence, the two functions $(\Pi, q) \mapsto D_q(\Pi), R_q(\Pi)$ are upper semicontinuous.*

Proof. Consider any convergent sequence $(\Pi_k, q_k)_k$ converging to (Π_∞, q_∞), and take $Y \in \{D, R\}$. Let ψ_Y be as given by Lemma 1, and note that the bounded function $(\theta, p) \mapsto \mathbf{1}_{p\leq u(\theta,q)}\psi_Y(p)$ is upper semicontinuous for any $q \in [0, 1]$ because u is upper semicontinuous.

For any $\hat{q} \in (q_\infty, 1) \cup \{1\}$, any large enough k has $q_k \leq \hat{q}$. Therefore,

$$\limsup_{k\to\infty} Y_{q_k}(\Pi_k) \leq \limsup_{k\to\infty} Y_{\hat{q}}(\Pi_k) \leq Y_{\hat{q}}(\Pi_\infty),$$

where the first inequality follows from monotonicity (Lemma 2) and the second from the portmanteau theorem.

If $q_\infty < 1$, then Lemma 2 tells us $Y_{\hat{q}}(\Pi_\infty)$ converges to $Y_{q_\infty}(\Pi_\infty)$ as $\hat{q} \searrow q_\infty$. Therefore, $\limsup_{k\to\infty} Y_{q_k}(\Pi_k) \leq Y_{q_\infty}(\Pi_\infty)$ as required. *Q.E.D.*

Remark 5. The previous three lemmas make use of the fact that u is (weakly) increasing in its second argument, bounded, and continuous; but they do not make use of our additional smoothness assumptions or concavity assumptions. Largely for this reason, their proofs apply (mutatis mutandis) to the complete information benchmark. In contrast, the subsequent results of this section make use of the additional smoothness enjoyed by our main model.

Lemma 4. *For any $\Pi \in \Delta(\mathbb{R}_+)$, the function $q \mapsto R_q(\Pi)$ is strictly increasing in the range where it is strictly positive.*

Proof. Suppose $q, \hat{q} \in [0,1]$ have $q < \hat{q}$ and $R_q(\Pi) > 0$ (implying $q > 0$). We want to show $R_{\hat{q}}(\Pi) > R_q(\Pi)$. To that end, observe that strict monotonicity of $\overline{v}$ and Assumption 1 together imply $D_{\hat{q}}(p) > D_q(p)$ for every $p \in (0, \overline{v}(\hat{q}))$. Because Π puts positive mass on this range of prices and $D_{\hat{q}} \geq D_q$ globally, it follows that $R_{\hat{q}}(\Pi) - R_q(\Pi) \geq \int_0^{\overline{v}(q)} p \left[D_{\hat{q}}(p) - D_q(p)\right] \, \mathrm{d}\Pi(p) > 0$. *Q.E.D.*

Lemma 5. *For any $\Pi \in \Delta(\mathbb{R}_+)$, the function $q \mapsto D_q(\Pi)$ is continuous.*

Proof. First, let us see the map $[0,1] \to \Delta(\mathbb{R}_+)$ taking any q to its associated distribution of willingness to pay (i.e., that with cumulative distribution function F_q) is continuous with respect to weak convergence of measures. Indeed, for any continuous bounded function $\psi : \mathbb{R}_+ \to \mathbb{R}$, the function

$$q \mapsto \int \psi \, \mathrm{d}\left[G \circ u(\cdot, q)^{-1}\right] = \int \psi \circ u(\cdot, q) \, \mathrm{d}G$$

is continuous because $\psi \circ u$ is, and so it converges to $\int \psi \, \mathrm{d}G \circ u(\cdot, \hat{q})^{-1}$ as $q \to \hat{q}$.

Now, observe that any $p \in \mathbb{R}_+$ has $q \mapsto D_q(p)$ continuous. First, this map is continuous at any $\hat{q} \in (0,1]$ by the observation of the above paragraph: Because $F_{\hat{q}}$ is atomless, the F_q-measure of the continuity set $[p, \infty)$ converges to the $F_{\hat{q}}$ of the same as $q \to \hat{q}$. Second, any sufficinetly small $q \in (0,1]$ has $D_q(p) = D_0(p)$—which is equal to 1 if $p = 0$ and equal to 0 otherwise.

Finally, given the previous paragraph, the Lebesgue dominated convergence theorem tells us $q \mapsto D_q(\Pi)$ is continuous. *Q.E.D.*

B. Proofs for Section 3

B.1. Proof of Proposition 1

Toward showing this program's solutions are exactly the optimal pairs (Π^*, q^*), let us invest in some terminology. Say a pair $(\Pi, q) \in \Delta(\mathbb{R}_+) \times [0,1]$ is *worst-feasible* if q is a worst equilibrium for the seller given price distribution Π. Say a pair $(\Pi^*, q^*) \in \Delta(\mathbb{R}_+) \times [0,1]$ is *limit-worst-feasible (LWF)* if it is a limit of a sequence of worst-feasible pairs. Finally, let $R := \sup_{(\Pi,q) \text{ worst-feasible}} R_q(\Pi)$ denote the seller's optimal value, that is, the optimal value of program (P).

We proceed by showing each of the following facts:

(i) Every LWF pair (Π, q), as witnessed by sequence $(\Pi_k, q_k)_k$, is feasible in program (P*) and has $\limsup_{k\to\infty} R_{q_k}(\Pi_k) \leq R_q(\Pi)$.[44]

(ii) If pair (Π, q) is feasible in program (P*), then some $\tilde{q} \geq q$ has $(\Pi, \tilde{q})$ LWF, as witnessed by a sequence $(\Pi_k, q_k)_k$ with $\liminf_{k\to\infty} R_{q_k}(\Pi_k) \geq R_q(\Pi)$.

(iii) The optimal value R^* of program (P*) is strictly positive.

(iv) Program (P*) admits an optimal solution.

(v) If (Π, q) is optimal in program (P*) and $(\Pi, \hat{q})$ is feasible in it, then $\hat{q} \leq q$.

Let us explain how the proposition would then follow. Because $\hat{q} \mapsto R_{\hat{q}}(\Pi)$ is increasing (Lemma 2), item (ii) tells us $R^* \leq R$; and item (i) tells us $R \leq R^*$, hence the two are equal. Item (i) then also implies that any worst-case optimal pair (Π, q) is also an optimum in program (P*); indeed, it must be feasible in (P*) and, if suboptimal in program (P*), will be LWF as witnessed by a sequence $(\Pi_k, q_k)_k$ with $\limsup_{k\to\infty} R_{q_k}(\Pi_k) < R^* = R$, yielding a contradiction. To see any optimum (Π, q) in program (P*) is worst-case optimal, note that (ii) delivers some $\tilde{q} \geq q$ with $(\Pi, \tilde{q})$ LWF as witnessed by a sequence $(\Pi_k, q_k)_k$ with $\lim_{k\to\infty} R_{q_k}(\Pi_k) = R^* = R$—thus, $(\Pi, \tilde{q})$ is worst-case optimal. But then (i) says $(\Pi, \tilde{q})$ is feasible in program (P*), and so (v) says $(\Pi, q) = (\Pi, \tilde{q})$ is

[44] In fact, the inequality holds with equality. However, only this inequality is required for the present proposition, enabling the result to extend to contingent price distributions for the complete information benchmark—see Proposition 2.

worst-case optimal. The proposition's last assertion then follows directly from (iii) and (iv). We now establish each of these five items.

Toward (i), let (Π, q) be any LWF pair, as witnessed by sequence $(\Pi_k, q_k)_k$. Lemma 3 tells us $\limsup_{k\to\infty} R_{q_k}(\Pi_k) \leq R_q(\Pi)$, and that any $\hat{q} \in (0, q)$ has $\limsup_{k\to\infty} D_{\hat{q}}(\Pi_k) \leq D_{\hat{q}}(\Pi)$. But given such $\hat{q}$, any large enough k has $q_k > \hat{q}$; that q_k is the smallest equilibrium quantity for Π_k then implies (by the Knaster-Tarski theorem) that $D_{\hat{q}}(\Pi_k) > \hat{q}$. Hence, $D_{\hat{q}}(\Pi) \geq \limsup_{k\to\infty} D_{\hat{q}}(\Pi_k) \geq \hat{q}$, meaning (Π, q) is feasible in program (P*).

To see (ii), consider a feasible pair (Π, q) in program (P*). For each $\varepsilon \in (0,1)$, the price distribution $\Pi^{\varepsilon} := (1-\varepsilon)\Pi + \varepsilon \mathbf{1}_{[0,\infty)}$ has $D_{\hat{q}}(\Pi^{\varepsilon}) \geq (1-\varepsilon)\hat{q} + \varepsilon > \hat{q}$ for every $\hat{q} \in [0, q)$, and so the smallest equilibrium quantity q^{ε} for this distribution (which exists by Lemma 2) has $q^{\varepsilon} \geq q$. By compactness of $[q, 1]$, some sequence of such ε converging to zero has q^{ε} converging to some $\tilde{q}$. Because $\hat{q} \mapsto R_{\hat{q}}(\Pi)$ yields $R_{q^{\varepsilon}}(\Pi^{\varepsilon}) = (1-\varepsilon) R_{q^{\varepsilon}}(\Pi) \geq (1-\varepsilon) R_q(\Pi) \xrightarrow{\varepsilon\to 0} R_q(\Pi)$ (where the inequality follows from Lemma 2), item (ii) follows.

For (iii), consider the price distribution Π that offers zero price to half of the buyers and price $\frac{1}{2}\overline{v}\left(\frac{1}{2}\right) \in \left(0, \overline{v}\left(\frac{1}{2}\right)\right)$ to the rest. Any $\hat{q} \in \left[0, \frac{1}{2}\right]$ has $D_{\hat{q}}(\Pi) > \hat{q}$, and so the worst-case equilibrium quantity q (which exists by Lemma 2) yields strictly positive revenue. The pair (Π, q) is then feasible in program (P*) and yields strictly positive revenue.

To see (iv), first note that for any $\Pi \in \Delta(\mathbb{R}_+)$, the capped price distribution $\tilde{\Pi} \in \Delta\,[0, \overline{v}(1)]$ given by $\tilde{\Pi}(p) := \Pi(p)\mathbf{1}_{p<\overline{v}(1)} + \mathbf{1}_{p\geq\overline{v}(1)}$ has $D_q(\tilde{\Pi}) = D_q(\Pi)$ and $R_q(\tilde{\Pi}) = R_q(\Pi)$ for every $q \in [0,1]$. It therefore suffices to see that the program restricted to price distributions in $\Delta\,[0, \overline{v}(1)]$ admits an optimum. But this program has a compact domain and (by Lemma 3) an upper semicontinuous objective, delivering the result.

Finally, we turn to (v). Suppose (Π, q) and $(\Pi, \hat{q})$ are both feasible in program (P*) and $\hat{q} > q$; we want to show (Π, q) cannot be optimal in (P*). If $R_q(\Pi) = 0$, then (Π, q) is not optimal by (iv). And if $R_q(\Pi) > 0$, then Lemma 4 says $R_q(\Pi) < R_{\hat{q}}(\Pi)$, again making pair (Π, q) suboptimal. *Q.E.D.*

B.2. Inputs for the proof of Theorem 1

The following lemma records a useful technical result that generalizes Proposition 4 of Rappoport (2025).

Lemma 6. *Suppose* $[\underline{v}, \overline{v}] \subset \mathbb{R}$ *is a nondegenerate interval;* $f, g : [\underline{v}, \overline{v}] \to \mathbb{R}$ *are absolutely integrable functions; and* $\psi : [\underline{v}, \overline{v}] \to \mathbb{R}$ *is a function of bounded variation.*[45]

(i) Suppose g *is zero wherever* f *is zero on* $[\underline{v}, \overline{v}]$*, and the ratio* $\frac{g}{f}$ *is weakly increasing where its denominator is nonzero. If* $\int_{\underline{v}}^{v} \psi f \geq 0$ *for every* $v \in [\underline{v}, \overline{v}]$*, with equality at* $v = \overline{v}$*, then* $\int_{\underline{v}}^{\overline{v}} \psi g \leq 0$.

(ii) Suppose g *is zero wherever* f *is zero on* $[\underline{v}, \overline{v}]$*, and the ratio* $\frac{g}{f}$ *is weakly increasing where its denominator is nonzero. If* $\int_{\underline{v}}^{v} \psi f \geq 0$ *for every* $v \in [\underline{v}, \overline{v}]$*, with equality at* $v = \overline{v}$*, and some* $v \in [\underline{v}, \overline{v}]$ *exists such that* $\int_{\underline{v}}^{v} \psi f > 0$ *and* $\frac{g}{f}$ *is not constant on any neighborhood of* v*, then* $\int_{\underline{v}}^{\overline{v}} \psi g < 0$.

(iii) Suppose f *is zero wherever* g *is zero on* $[\underline{v}, \overline{v}]$*, and the ratio* $\frac{f}{g}$ *is weakly decreasing where its denominator is nonzero. If* $f(\overline{v}), g(\overline{v}) \geq 0$ *and* $\int_{\underline{v}}^{v} \psi g \geq 0$ *for every* $v \in [\underline{v}, \overline{v}]$*, then* $\int_{\underline{v}}^{\overline{v}} \psi f \geq 0$.

Proof. First, we prove parts (i) and (ii). To that end, suppose the hypotheses of part (i) are satisfied. In what follows, we interpret $\frac{g}{f}$ as an arbitrary nondecreasing function $[\underline{v}, \overline{v}] \to \mathbb{R}$ that agrees with $\frac{g}{f}$ wherever f is nonzero. Now, define the absolutely continuous function $\Psi : [\underline{v}, \overline{v}] \to \mathbb{R}$ by letting $\Psi(v) := \int_{\underline{v}}^{v} \psi f$. Then, we can apply integration by parts for Stieltjes integra-

[45] Throughout, for any interval $[\underline{v}, \overline{v}] \subseteq \mathbb{R}$ and any Lebesgue integrable function $h : [\underline{v}, \overline{v}] \to \mathbb{R}$, we let $\int_{\underline{v}}^{\overline{v}} h$ denote the Lebesgue integral $\int_{\underline{v}}^{\overline{v}} h(v) \, \mathrm{d}v$.

tion:

$$\begin{aligned}
\int_{\underline{v}}^{\overline{v}} \psi g &= \int_{\underline{v}}^{\overline{v}} \tfrac{g}{f}\Psi' \\
&= \left[\Psi \tfrac{g}{f}\right]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} \Psi \ \mathrm{d}\tfrac{g}{f} \text{ (by integration by parts)} \\
&= 0 - \int_{\underline{v}}^{\overline{v}} \Psi \ \mathrm{d}\tfrac{g}{f} \text{ (since } \Psi(\underline{v}) = \Psi(\overline{v}) = 0) \\
&\leq 0 \text{ (since } \Psi \geq 0 \text{ and } \tfrac{g}{f} \text{ is weakly increasing)},
\end{aligned}$$

establishing part (i). Now, suppose in addition that some $v \in [\underline{v}, \overline{v}]$ exists such that $\Psi(v) > 0$ and $\frac{g}{f}$ is not constant on any neighborhood of v. By continuity, Ψ is strictly positive on some some nondegenerate interval of v. Because $\frac{g}{f}$ is not constant on this interval, it follows that $\int_{\underline{v}}^{\overline{v}} \psi g = -\int_{\underline{v}}^{\overline{v}} \Psi \ \mathrm{d}\frac{g}{f} < 0$, delivering (ii).

Next, we prove part (iii); suppose its hypotheses are satisfied. In what follows, we interpret $\frac{f}{g}$ as an arbitrary nonincreasing function $[\underline{v}, \overline{v}] \to \mathbb{R}$ that agrees with $\frac{f}{g}$ wherever g is nonzero.

Now, define the absolutely continuous functions $\Phi : [\underline{v}, \overline{v}] \to \mathbb{R}$ by letting $\Phi(v) := \int_{\underline{v}}^{v} \psi g$. Then,

$$\begin{aligned}
\int_{\underline{v}}^{\overline{v}} \psi f &= \int_{\underline{v}}^{\overline{v}} \tfrac{f}{g}\Phi' \\
&= \left[\Phi \tfrac{f}{g}\right]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} \Phi \ \mathrm{d}\tfrac{f}{g} \text{ (by integration by parts)} \\
&= \Phi(\overline{v})\tfrac{f(\overline{v})}{g(\overline{v})} - \int_{\underline{v}}^{\overline{v}} \Phi \ \mathrm{d}\tfrac{f}{g} \text{ (since } \Phi(\underline{v}) = 0) \\
&\geq \Phi(\overline{v})\tfrac{f(\overline{v})}{g(\overline{v})} \text{ (since } \Phi \geq 0 \text{ and } \tfrac{f}{g} \text{ is weakly decreasing)} \\
&\geq 0 \text{ (since } \Phi(\overline{v}), f(\overline{v}), g(\overline{v}) \geq 0),
\end{aligned}$$

as required. *Q.E.D.*

The following lemma is a comparative statics result for comparing different price distributions: if a reduction in price dispersion preserves aggregate demand under a low anticipated quantity (and the only modified prices are those that will sometimes be exercised), then the reduction increases both demand and revenue when the anticipated quantity is higher.

Lemma 7. *Given $q \in (0,1]$, suppose distinct price distributions $\Pi, \tilde{\Pi} \in \Delta(\mathbb{R}_+)$ are such that $\Pi|_{(\overline{v}(q),\infty)} = \tilde{\Pi}|_{(\overline{v}(q),\infty)}$, and*

$$\int_0^v (\Pi - \tilde{\Pi}) f_q \geq 0$$

for every $v \in [0, \overline{v}(q)]$, with equality at $v = \overline{v}(q)$.[46] *Then, any $\tilde{q} \in [q, 1]$ has*

$$D_{\tilde{q}}(\tilde{\Pi}) \geq D_{\tilde{q}}(\Pi) \text{ and } R_{\tilde{q}}(\tilde{\Pi}) > R_{\tilde{q}}(\Pi).$$

Proof. Both rankings can be derived as applications of Lemma 6, with $(\underline{v}, \overline{v}, f, \psi) = (0, \overline{v}_q, f_q, \Pi - \tilde{\Pi})$ and different choices of g.

First, consider $g := f_{\tilde{q}}|_{[0,\overline{v}(q)]}$, and apply Assumption 1. By Lemma 6(i),[47]

$$0 \leq \int_0^{\overline{v}(q)} (\tilde{\Pi} - \Pi) f_{\tilde{q}} = \int_0^{\overline{v}(\tilde{q})} (\tilde{\Pi} - \Pi) f_{\tilde{q}} = D_{\tilde{q}}(\tilde{\Pi}) - D_{\tilde{q}}(\Pi).$$

Next, consider $g := \varphi_{q,\tilde{q}} f_q$. As Assumption 3 holds, Lemma 6(ii) tells us

$$\begin{aligned} 0 &< \int_0^{\overline{v}(q)} (\tilde{\Pi} - \Pi)\varphi_{q,\tilde{q}} f_q = \int_0^{\overline{v}(\tilde{q})} (\tilde{\Pi} - \Pi)\varphi_{\tilde{q},\tilde{q}} \, \mathrm{d}F_{\tilde{q}} \\ &= \int_0^{\overline{v}(\tilde{q})} (\Pi - \tilde{\Pi}) \, \mathrm{d}R_{\tilde{q}} = 0 - \int R_{\tilde{q}} \, \mathrm{d}(\Pi - \tilde{\Pi}) \\ &= R_{\tilde{q}}(\tilde{\Pi}) - R_{\tilde{q}}(\Pi). \end{aligned}$$

Q.E.D.

The following lemma extends our concave externalities assumption to measures over prices rather than just prices.

[46] Note, the equality at $\overline{v}(q)$ says exactly that $D_q(\tilde{\Pi}) = D_q(\Pi)$.

[47] One can alternatively prove this ranking by using the fact that $F_{\tilde{q}} \circ F_q^{-1}$ is convex under Assumption 1.

Lemma 8. *Suppose* $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *is right-continuous, nondecreasing, and bounded, and* $0 \le q_0 < q_1 \le 1$ *have* $\Gamma(\overline{v}(q_0)) = \Gamma(\overline{v}(q_1)^-)$. *Then* $q \mapsto D_q(\Gamma)$ *is concave on* $[q_0, q_1]$, *strictly so if* $\Gamma(\overline{v}(q_0)) > 0$.

Proof. For any price $p \le \overline{v}(q_0)$, the function $q \mapsto D_q(p)$ is strictly concave on (q_0, q_1) by Assumption 2, hence on $[q_0, q_1]$ by Lemma 5. For any price $p \ge \overline{v}(q_1)$, the function $q \mapsto D_q(p)$ is zero on $[q_0, q_1]$. Because a pointwise weighted sum of concave functions is concave, strictly so if this sum puts strictly positive weight on strictly concave functions, the lemma follows. *Q.E.D.*

To state the next lemma, we invest in some notation.

Notation 1.

- *Let* $\dot{f}_q(v)$ *denote the partial derivative of* $f_q(v)$ *with respect to* q, *which exists wherever* $q \in (0, 1]$ *and* $0 \le v \le \overline{v}(q)$.
- *Let* $\partial D_q(\Pi)$ *[resp.* $\partial^- D_q(\Pi)$ *or* $\partial^+ D_q(\Pi)$*] denote the partial derivative [resp. left derivative or right derivative] of* $D_q(\Pi)$ *with respect to* q, *if it exists.*

The following lemma establishes that one-sided derivatives of demand with respect to anticipated quantity are finite, and that the demand function is kinked if and only if the price distribution has a mass point.

Lemma 9. *Suppose* $\Gamma : [0, \overline{v}(1)) \to \mathbb{R}_+$ *is increasing and right continuous, and* $q \in (0, 1]$. *Then:*

- $\partial^- D_q(\Gamma) = \int_0^{\overline{v}(q)} \Gamma \dot{f}_q + \overline{v}'(q)\Gamma(\overline{v}(q)^-) f_q(\overline{v}(q)) \in \mathbb{R}$.
- *If* $q < 1$, *then* $\partial^+ D_q(\Pi) = \int_0^{\overline{v}(q)} \Gamma \dot{f}_q + \overline{v}'(q)\Gamma(\overline{v}(q)) f_q(\overline{v}(q)) \in \mathbb{R}$.
- *If* Γ *is continuous at* $\overline{v}(q)$, *then* $\tilde{q} \mapsto D_{\tilde{q}}(\Gamma)$ *is differentiable at* q.
- *If* $q < 1$ *and* Γ *is discontinuous at* $\overline{v}(q)$, *then* $\tilde{q} \mapsto D_{\tilde{q}}(\Gamma)$ *has a convex kink at* q.

Proof. Whenever $0 \leq q_0 < q_1 \leq 1$, we have

$$\begin{aligned}\tfrac{D_{q_1}(\Gamma)-D_{q_0}(\Gamma)}{q_1-q_0} &= \tfrac{1}{q_1-q_0}\left[\int_0^{\overline{v}(q_1)} \Gamma f_{q_1} - \int_0^{\overline{v}(q_0)} \Gamma f_{q_0}\right] \\ &= \int_0^{\overline{v}(q_0)} \Gamma \tfrac{f_{q_1}-f_{q_0}}{q_1-q_0} + \tfrac{\overline{v}(q_1)-\overline{v}(q_0)}{q_1-q_0} \tfrac{1}{\overline{v}(q_1)-\overline{v}(q_0)} \int_{\overline{v}(q_0)}^{\overline{v}(q_1)} \Gamma f_{q_1}.\end{aligned}$$

Given the Lebesgue dominated convergence theorem, the first two points come from applying this expression as $q_0 \nearrow q = q_1$ and as $q_1 \searrow q = q_0$, respectively. Then, combine the first two points for $q \in (0,1)$ to obtain

$$\partial^+ D_q(\Gamma) - \partial^- D_q(\Gamma) = \overline{v}'(q) f_q(\overline{v}(q)) \left[\Gamma\left(\overline{v}(q)\right) - \Gamma\left(\overline{v}(q)^-\right)\right],$$

directly implying the last two points. *Q.E.D.*

B.3. Proof of Theorem 1

We begin with some useful terminology.

Definition 4. *Consider any price distribution* Π. *Given* $q \in [0,1]$*:*

- *Say* Π ***has mass at*** q^{++} *if* $\Pi(p) > \Pi\left(\overline{v}(q)\right)$ *for every* $p > \overline{v}(q)$.
- *Say* Π ***has mass at*** q^{--} *if* $\Pi(p) < \Pi\left(\overline{v}(q)^-\right)$ *for every* $p < \overline{v}(q)$.
- *Say* Π ***has mass at*** q^+ *[resp.* q^-*] if it has a mass at* q^{++} *[resp. at* q^{--}*] or has a mass point at* $\overline{v}(q)$.

Given $q_0, q_1 \in [0,1]$ *with* $q_0 < q_1$, *say* Π *is* ***degenerate on*** $[q_0, q_1]$ *if some* $p \in [\overline{v}(q_0), \overline{v}(q_1)]$ *exists such that* $\Pi\left(p^-\right) = \Pi\left(\overline{v}(q_0)^-\right)$ *and* $\Pi\left(p\right) = \Pi\left(\overline{v}(q_1)\right)$.

The following claim shows any optimal price distribution in the subproblem associated with any targeted quantity uses only prices below the monopoly price for that anticipated quantity's demand curve.

Claim 1. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $\hat{q} \in (0,1]$ *have* $\Pi\left(p^M(\hat{q})\right) < 1$. *Then, some* $\tilde{\Pi} \in \Delta(\mathbb{R}_+)$ *exists such that* $D_q(\tilde{\Pi}) \geq D_q(\Pi)$ *for every* $q \in [0,1]$, *and* $R_{\hat{q}}(\tilde{\Pi}) > R_{\hat{q}}(\Pi)$.

Proof. Let $p^* := p^M(\hat{q})$, and let $\tilde{\Pi} := \Pi|_{[0,p^*)} \cup \mathbf{1}|_{[p^*,\infty)}$. The distribution $\tilde{\Pi}$ is below Π in the sense of first-order stochastic dominance, so that $D_q(\tilde{\Pi}) \geq D_q(\Pi)$ for every $q \in [0,1]$. Moreover, Assumption 3 implies any price $p \neq p^*$ has $R_{\hat{q}}(p) < R_{\hat{q}}(p^*)$. Therefore, given that $\Pi(p^*) < 1$, we have

$$R_{\hat{q}}(\tilde{\Pi}) - R_{\hat{q}}(\Pi) = \int_{p^*}^{\infty} \left[R(p^*) - R(p)\right] \, \mathrm{d}\Pi(p) > 0,$$

as desired. *Q.E.D.*

The following claim uses concave externalities to establish that the slack on the demand constraint is first-order wherever the price distribution has a gap at the edge of a slack region.

Claim 2. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $q \in [0,1]$ *have* $D_q(\Pi) = q$*.*

- *If* $q < 1$*, every* $\tilde{q} > q$ *close enough to* q *has* $D_{\tilde{q}}(\Pi) > \tilde{q}$*, and* Π *has no mass at* q^{++}*, then* $\partial^+ D_q(\Pi) > 1$*.*
- *If* $q > 0$*, every* $\tilde{q} < q$ *close enough to* q *has* $D_{\tilde{q}}(\Pi) > \tilde{q}$*, and* Π *has no mass at* q^{--}*, then* $\partial^- D_q(\Pi) < 1$*.*

Proof. Define the function $\psi : [0,1] \to \mathbb{R}$ via $\psi(\tilde{q}) := D_{\tilde{q}}(\Pi) - \tilde{q}$, which is continuous by Lemma 5. By Lemma 8, we know ψ is concave in an interval to the right [resp. left] of q if $q < 1$ [resp. $q > 0$] and Π has no mass at q^{++} [resp. q^{--}].

Now, if ψ is zero at q and concave and strictly positive in a right [resp. left] neighborhood of q, it follows that its right [resp. left] derivative at q is strictly positive [resp. strictly negative], delivering the claim. *Q.E.D.*

The following claim shows that a feasible price distribution is always non-degenerate over (the closure of) any slack region in the range of its support.

Claim 3. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $p^* := \max \operatorname{supp} \Pi$ *has* $D_q(\Pi) \geq q$ *for every* $q \in (0, \underline{q}(p^*))$*. If* (q_0, q_1) *is a connected component of*

$$\left\{q \in \left(0, \underline{q}(p^*)\right) : \; D_q(\Pi) > q\right\},$$

then Π *is nondegenerate on* $[q_0, q_1]$.

Proof. The claim holds vacuously if $p^* = 0$, so focus on the case in which $p^* > 0$.

If Π has mass at q_0^{++} or at q_1^{--}, it is clearly nondegenerate on $[q_0, q_1]$. So now, focus on the case in which Π has mass neither at q_0^{++} nor at q_1^{--}. The claim will now follow if we establish that Π has mass points both at $\overline{v}(q_0)$ and at $\overline{v}(q_1)$.

Observe first that $\min \operatorname{supp} \Pi = 0$, for otherwise small enough $q \in \big(0, \underline{q}(p^*)\big)$ will have $D_q(\Pi) = 0 < q$. Then, by definition of the support (and the hypothesis that Π has mass neither at q_0^{++} nor at q_1^{--}), we know that Π has a mass point at $0 = \overline{v}(q_0)$ if $q_0 = 0$, and has a mass point at $p^* = \overline{v}(q_1)$ if $q_1 = \underline{q}(p^*)$.

It remains now to show that Π has a mass point at $\overline{v}(q_0)$ if $q_0 > 0$, and has a mass point at $\overline{v}(q_1)$ if $q_1 < \underline{q}(p^*)$. So suppose $q_0 > 0$ [resp. $q_1 < \underline{q}(p^*)$]. By definition of (q_0, q_1), no $\tilde{q}_0 < q_0$ [resp. $\tilde{q}_1 > q_1$] exists such that every $q \in [\tilde{q}_0, q_0]$ [resp. every $q \in [q_1, \tilde{q}_1]$] has $D_q(\Pi) > q$. But then, by Lemma 5 we in fact have that $D_{q_0}(\Pi) = q_0$ [resp. $D_{q_1}(\Pi) = q_1$]. Claim 2 thus implies $\partial^+ D_{q_0}(\Pi) > 1$ [resp. $\partial^- D_{q_1}(\Pi) < 1$]. Meanwhile, that $q \mapsto D_q(\Pi) - q$ is zero at q_0 [resp. q_1] and nonnegative just to the left [resp. right] of it implies $\partial^- D_{q_0}(\Pi) \leq 1$ [resp. $\partial^+ D_{q_1}(\Pi) \geq 1$]. Thus, $q \mapsto D_q(\Pi)$ has a convex kink at q_0 [resp. q_1], and so Lemma 9 tells us Π has a mass point at $\overline{v}(q_0)$ [resp. $\overline{v}(q_1)$] as desired. *Q.E.D.*

The following claim says that whenever the price distribution is nondegenerate over some interval, a smaller such interval can be found on which the price distribution is also well-behaved.

Claim 4. *Suppose* $\Pi \in \Delta(\mathbb{R}_+)$ *and* $0 \leq q_0 < q_1 \leq 1$ *are such that* Π *is nondegenerate on* $[q_0, q_1]$. *Then some* $\tilde{q}_0, \tilde{q}_1 \in [q_0, q_1]$ *with* $\tilde{q}_0 < \tilde{q}_1$ *exist such that:*

- Π *is nondegenerate on* $[\tilde{q}_0, \tilde{q}_1]$*;*
- *either* $\tilde{q}_0 \in (q_0, q_1)$ *or* Π *has no mass at* q_0^{++}*;*
- *either* $\tilde{q}_1 \in (q_0, q_1)$ *or* Π *has no mass at* q_1^{--}.

Proof. If Π has mass at q_1^{--}, then any $\tilde{q}_0 \in (q_0, q_1)$, paired with any $\tilde{q}_1 \in (\tilde{q}_0, q_1)$ close enough to q_1, is as desired. If Π has mass at q_0^{++}, then any $\tilde{q}_1 \in (q_0, q_1)$, paired with any $\tilde{q}_0 \in (q_0, \tilde{q}_1)$ close enough to q_0, is as desired. If Π has no mass at q_1^{--} or at q_0^{++}, then $\tilde{q}_0 = q_0$ and $\tilde{q}_1 = q_1$ are as desired. *Q.E.D.*

The following claim shows that small enough perturbations preserve the demand constraint on any interval where it is slack (with first-order slack at the edges).

Claim 5. *Suppose* $\Pi, \tilde{\Pi} \in \Delta(\mathbb{R}_+)$ *and* $0 \le q_0 < q_1 \le 1$ *are such that:*

- *Every* $q \in (q_0, q_1)$ *has* $D_q(\Pi) > q$*, and each* $q \in \{q_0, q_1\}$ *has* $D_q(\tilde{\Pi}) \ge q$*;*
- *Either* $D_{q_0}(\Pi) > q_0$ *or* $\partial^+ D_{q_0}(\Pi) > 1$*, with the former case if* $q_0 = 0$*; and*
- *Either* $D_{q_1}(\Pi) > q_1$ *or* $\partial^- D_{q_1}(\Pi) < 1$*.*

Then, letting $\Pi_\varepsilon := (1-\varepsilon)\Pi + \varepsilon\tilde{\Pi}$*, any small enough* $\varepsilon \in (0,1)$ *has*

$$D_q(\Pi_\varepsilon) \ge q, \ \forall q \in [q_0, q_1].$$

Proof. First, for either $q \in \{q_0, q_1\}$, if $D_q(\Pi) > q$, then (given that $D_q(\tilde{\Pi}) \ge q$) every $\varepsilon \in (0,1)$ has $D_q(\Pi_\varepsilon) > q$. Next, if either $q \in \{q_0, q_1\}$ has $D_q(\Pi) = q$ (which in particular means $q > 0$ given our hypotheses), then Lemma 9 tells us one-sided derivatives of $\tilde{q} \mapsto D_{\tilde{q}}(\tilde{\Pi})$ are finite there. So, for small enough $\bar{\varepsilon} \in (0,1)$:

- Each $q \in \{q_0, q_1\}$ has $D_q(\Pi_{\bar{\varepsilon}}) \ge q$;
- Either $D_{q_0}(\Pi_{\bar{\varepsilon}}) > q_0$ or $\partial^+ D_{q_0}(\Pi_{\bar{\varepsilon}}) > 1$; and
- Either $D_{q_1}(\Pi_{\bar{\varepsilon}}) > q_1$ or $\partial^- D_{q_1}(\Pi_{\bar{\varepsilon}}) < 1$.

Fixing such an $\bar{\varepsilon}$, some $\tilde{q}_0, \tilde{q}_1 \in (q_0, q_1)$ exist such that every $q \in (q_0, \tilde{q}_0] \cup [\tilde{q}_1, q_1)$ has $D_q(\Pi_{\bar{\varepsilon}}) > q$. Hence, because $\varepsilon \mapsto D_q(\Pi_\varepsilon)$ is affine for every q, it follows that every $q \in (q_0, \tilde{q}_0] \cup [\tilde{q}_1, q_1)$ and $\varepsilon \in (0, \bar{\varepsilon}]$ have $D_q(\Pi_\varepsilon) \ge q$.

Hence, all that remains is to see (focusing on the nontrivial case that $q_0 < q_1$) that sufficiently small $\varepsilon \in (0, \bar{\varepsilon}]$ have $D_q(\Pi_\varepsilon) \ge q$ for every $(\tilde{q}_0, \tilde{q}_1)$.

And indeed, given Lemma 5, Berge's theorem tells us the function $[0, \overline{\varepsilon}] \to \mathbb{R}$ given by $\varepsilon \mapsto \min_{q \in [\tilde{q}_0, \tilde{q}_1]} [D_q(\Pi_\varepsilon) - q]$ is well-defined and continuous. Because $[\tilde{q}_0, \tilde{q}_1] \subset (q_0, q_1)$, this function is strictly positive at $\varepsilon = 0$, and so is strictly positive for small enough $\varepsilon \in (0, \overline{\varepsilon}]$, delivering the claim. *Q.E.D.*

Now, with these claims in hand, we pursue the proof of the theorem.

Proof of Theorem 1. First, given Claim 1, any optimal (Π^*, q^*) must have $\max \operatorname{supp} \Pi^* \leq p^M(q^*)$.

Now, we show that any optimal (Π^*, q^*) has Π^* greedy up to the top of its support. To that end, consider $q^* \in [0, 1]$ and $\Pi \in \Delta(\mathbb{R}_+)$ such that $D_q(\Pi) \geq q$ for every $q \in (0, q^*)$, and Π is not greedy up to $p^* := \max \operatorname{supp} \Pi$. We want to show (Π, q^*) cannot be optimal. We have nothing to show (given the previous paragraph) if $p^* \geq \overline{v}(q^*)$, so without loss say $p^* < \overline{v}(q^*)$. Now, by hypothesis, the set

$$\left\{ q \in \big(0, \underline{q}(p^*)\big) : \ D_q(\Pi) > q \right\}$$

is nonempty. Meanwhile, Lemma 5 implies this set is open in $\mathbb{R}$, and so every connected component of it is an open interval. Let (q_0, q_1) be such a connected component. Claim 3 (which applies because $p^* \leq \overline{v}(q^*)$) tells us Π is nondegenerate on $[q_0, q_1]$. Hence, Claim 4 delivers some $\tilde{q}_0, \tilde{q}_1 \in [q_0, q_1]$ with $\tilde{q}_0 < \tilde{q}_1$ such that Π is nondegenerate on $[\tilde{q}_0, \tilde{q}_1]$; either $\tilde{q}_0 \in (q_0, q_1)$ or Π has no mass at q_0^{++}; and either $\tilde{q}_1 \in (q_0, q_1)$ or Π has no mass at q_1^{--}. Moreover, by Lemma 5, we know $D_{q_0}(\Pi) \geq q_0$ and $D_{q_1}(\Pi) \geq q_1$. Hence, applying Claim 2, we therefore have that either $D_{\tilde{q}_0}(\Pi) > \tilde{q}_0$ or $\partial^+ D_{\tilde{q}_0}(\Pi) > 1$; and either $D_{\tilde{q}_1}(\Pi) > \tilde{q}_1$ or $\partial^- D_{\tilde{q}_1}(\Pi) < 1$. So given any $\tilde{\Pi} \in \Delta(\mathbb{R}_+)$ with $D_{\tilde{q}_0}(\tilde{\Pi}) \geq \tilde{q}_0$ and $D_{\tilde{q}_1}(\tilde{\Pi}) \geq \tilde{q}_1$, Claim 5 tells us sufficiently small $\varepsilon \in (0, 1)$ has $D_q\Big((1-\varepsilon)\Pi + \varepsilon\tilde{\Pi}\Big) \geq q$ for every $q \in [\tilde{q}_0, \tilde{q}_1]$.

We are now equipped to show (Π, q^*) is suboptimal. For any $p \in [\overline{v}(\tilde{q}_0), \overline{v}(\tilde{q}_1)]$, consider the price distribution Π^p which coincides with Π on $[0, \overline{v}(\tilde{q}_0)) \cup [\overline{v}(\tilde{q}_1), \infty)$, takes value $\Pi\left(\overline{v}(\tilde{q}_0)^-\right)$ on $[\overline{v}(\tilde{q}_0), p)$, and takes value $\Pi\left(\overline{v}(\tilde{q}_1)\right)$ on $[p, \overline{v}(\tilde{q}_1))$. Observe that $p \mapsto D_{q_1}(p)$ is decreasing, Π lies between $\Pi^{\overline{v}(\tilde{q}_0)}$ and $\Pi^{\overline{v}(\tilde{q}_1)}$ (in the sense of first-order stochastic dominance), and $p \mapsto D_{q_1}(\Pi^p)$ is continuous because $p \mapsto D_{q_1}(p)$ is. Hence, the intermediate value theorem yields some $p \in [\overline{v}(\tilde{q}_0), \overline{v}(\tilde{q}_1)]$ such that $D_{q_1}(\Pi^p) = D_{q_1}(\Pi)$. For any $\varepsilon \in (0,1)$, let $\Pi_\varepsilon := (1-\varepsilon)\Pi + \varepsilon\Pi^p$. By construction, every $q \in (0, q_0]$ has $D_q(\Pi^p) = D_q(\Pi) \geq q$. Meanwhile Lemma 7 tells us $R_{q^*}(\Pi_\varepsilon) > R_{q^*}(\Pi)$ and every $q \in [q_1, q^*)$ has $D_q(\Pi^p) \geq D_q(\Pi) \geq q$. Therefore, for any $\varepsilon \in (0,1)$, we have $R_{q^*}(\Pi_\varepsilon) > R_{q^*}(\Pi)$ and $D_q(\Pi^p) \geq q$ for every $q \in (0, q_0] \cup [q_1, q^*)$. Finally, as noted in the previous paragraph, sufficiently small $\varepsilon \in (0,1)$ has $D_q(\Pi^p) \geq q$ for every $q \in [\tilde{q}_0, \tilde{q}_1]$. So (Π_ε, q^*) witnesses that (Π, q^*) is suboptimal, as claimed above.

Now, letting (Π^*, q^*) be optimal and $p^* := \max \operatorname{supp} \Pi^*$, we have established that Π^* is greedy up to p^* and $p^* \leq p^M(q^*)$. All that remains is to see Π^* has a mass point at p^*. To that end, note that $p^* \leq p^M(q^*) < \overline{v}(q^*)$ implies $\underline{q}(p^*) < q^*$. We can therefore apply Claim 2 to learn $\partial^+ D_{\underline{q}(p^*)}(\Pi^*) > 1$. But greediness up to p^* directly tells us $\partial^- D_{\underline{q}(p^*)}(\Pi^*) = 1$, and so Lemma 9 implies Π^* has a mass point at p^*. *Q.E.D.*

B.4. Proof of Corollary 1

Corollary 1 follows directly from Theorem 1 and the next Lemma 10, which shows that greediness rules out mass points and gaps in a price distribution.

Lemma 10. *Suppose $\hat{q} \in (0,1]$ and $\Gamma : [0, \overline{v}(\hat{q})) \to \mathbb{R}_+$ is increasing and right continuous with $D_q(\Gamma) = q$ for every $q \in (0, \hat{q})$. Then Γ is strictly increasing and continuous on $[0, \overline{v}(\hat{q}))$ with $\Gamma(0) = 0$.*

Proof. By hypothesis, $q \mapsto D_q(\Gamma)$ is differentiable on $(0, \hat{q})$, and so Lemma 9 tells us Γ has no discontinuities in $(0, \overline{v}(\hat{q}))$. Moreover, $\Gamma(0) = D_0(\Gamma) = 0$.

To show Γ is strictly increasing on $[0, \overline{v}(\hat{q}))$, it suffices to show (given that it is weakly increasing by definition and $\overline{v}$ is strictly increasing) that $\Gamma \circ \overline{v}$ is not constant over any interval. So suppose $0 < q_0 < q_1 < \overline{v}(\hat{q})$. Because $D_{q_0}(\Gamma) = q_0 > 0$, Lemma 8 would imply $q \mapsto D_q(\Gamma)$ is strictly concave if $\Gamma \circ \overline{v}$ were constant on (q_0, q_1). But this function is linear by hypothesis, hence not strictly concave. It follows that $\Gamma \circ \overline{v}$ is not constant on (q_0, q_1). *Q.E.D.*

B.5. Proof of Corollary 3

The following lemma will be useful in establishing the corollary.

Lemma 11. *With proportional values, the density g is (weakly) increasing.*

Before proving the lemma, let us prove the corollary taking the lemma for granted. To that end, suppose the environment has proportional values.

We first show that for any $\hat{p} > 0$, any two increasing and right-continuous functions $\mathbb{R}_+ \to \mathbb{R}_+$ that are greedy up to $\hat{p}$ must coincide on $[0, \hat{p})$. Let ψ be the difference of any two such functions, which has bounded variation by construction and is continuous on $[0, \hat{p})$ by Lemma 10. Fixing arbitrary $\hat{q} \in (0, \hat{p})$, we want to show $\psi|_{[0,\hat{q}]}$ is zero. It suffices to show it is nonpositive, since the same argument applies equally to $-\psi$. By hypothesis, $q \in [0, \hat{q}]$ have

$$0 = q0 = q \int_0^q \psi f_q = \int_0^q \psi(p) g\left(\tfrac{p}{q}\right) \, \mathrm{d}p,$$

and so differentiating with respect to q yields

$$\begin{aligned}
0 &= \psi(q)g(1) + \int_0^q \psi(p) g'\left(\tfrac{p}{q}\right)\left(\tfrac{-p}{q^2}\right) \, \mathrm{d}p \\
&= \psi(q)g(1) - \int_0^q \psi(p)\left(\tfrac{p}{q}\right) g'\left(\tfrac{p}{q}\right) \tfrac{\mathrm{d}p}{q} \\
&= \psi(q)g(1) - \int_0^1 \psi(q\theta)\theta g'(\theta) \, \mathrm{d}\theta \\
&= \psi(q)\left[g(1) - \int_0^1 \theta g'(\theta) \, \mathrm{d}\theta\right] + \int_0^1 \left[\psi(q) - \psi(q\theta)\right] \theta g'(\theta) \, \mathrm{d}\theta \\
&= \psi(q) \int_0^1 1 g(\theta) \, \mathrm{d}\theta + \int_0^1 \left[\psi(q) - \psi(q\theta)\right] \theta g'(\theta) \, \mathrm{d}\theta \\
&= \psi(q) + \int_0^1 \left[\psi(q) - \psi(q\theta)\right] \theta g'(\theta) \, \mathrm{d}\theta.
\end{aligned}$$

Because ψ is continuous, its restriction to the compact set $[0, \hat{q}]$ is maximized at some q. Applying the above equation at this q yields

$$0 = \psi(q) + \int_0^1 \left[\psi(q) - \psi(q\theta)\right] \theta g'(\theta) \, \mathrm{d}\theta \geq \psi(q),$$

where the inequality holds because $g' \geq 0$ by Lemma 11. Thus, $\psi|_{[0,\hat{q}]} \leq$

$\psi(q) \leq 0$, as desired.

Next, defining the average type $\hat{\theta} := \int_0^1 \theta g(\theta)\ \mathrm{d}\theta > 0$, define the function $\Gamma^* : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\Gamma^*(p) := \frac{1}{\hat{\theta}} p$. Let us observe that Γ^* is greedy. Because Theorem 1 tells us any optimal type distribution is greedy up the top of its support p^*, it will follow from the previous paragraph that any optimal price CDF assigns $\Gamma^*(p)$ to every price $p < p^*$. To see greediness, observe that each $q \in [0,1]$ has

$$\int_0^q \Gamma^* f_q = \int_0^q \tfrac{1}{\hat{\theta}} p g\left(\tfrac{p}{q}\right) \tfrac{1}{q}\ \mathrm{d}p = \tfrac{q}{\hat{\theta}} \int_0^q \tfrac{p}{q} g\left(\tfrac{p}{q}\right) \tfrac{\mathrm{d}p}{q} = \tfrac{q}{\hat{\theta}} \int_0^1 \theta g(\theta)\ \mathrm{d}\theta = q,$$

as required. *Q.E.D.*

The only unresolved detail for the corollary is Lemma 11.

Remark 6. As we will see, the proof of Lemma 11 invokes Assumption 1 and Assumption 3 but not Assumption 2. Because $q \mapsto p/q$ is strictly convex on $[p, 1]$ for every $p \in (0, 1]$, it follows from g being increasing that $q \mapsto G(p/q)$ is strictly convex too. Therefore, Assumption 2 is implied by the other concavity assumptions in the proportional values case. This feature does not hold in general, as apparent from the linear demand case.

We now proceed to establish the lemma.

Proof of Lemma 11. For convenience, consider the transformed PDF $\tilde{g} : \mathbb{R}_- \to \mathbb{R}$ given by $\tilde{g}(t) := t + \log g(e^t)$. Let us make some observations about this transformed function that will be useful. First, a straightforward computation shows Assumption 1—which we noted in the main text amounts to $\theta \mapsto g(\theta)/g(\alpha\theta)$ being increasing for every $\alpha > 1$—is equivalently expressed as $\tilde{g}$ being concave. Next, note that any $t \leq 0$ has $e^t g(e^t) = e^{\tilde{g}(t)}$, implying

$$\int_{-\infty}^0 e^{\tilde{g}(t)}\ \mathrm{d}t = \int_{-\infty}^0 g(e^t) e^t\ \mathrm{d}t = \int_0^1 g(\theta)\ \mathrm{d}\theta = 1.$$

In particular, the function $e^{\tilde{g}} > 0$ is quasiconcave (and so its slope converges at $-\infty$) and integrable on $\mathbb{R}_-$. Thus, as $t \to -\infty$, its level $e^{\tilde{g}(t)}$ converges to zero, and so too does its slope $e^{\tilde{g}(t)} \tilde{g}'(t)$.

With the above computations in hand, we turn to showing g is increasing. Observe that (because exponentiation is strictly increasing) g is increasing if and only if $t \mapsto \tilde{g}(t) - t$ is, so our goal is to prove that $\tilde{g}' \geq 1$. Moreover, because $\tilde{g}$ is concave, it suffices to argue that $\tilde{g}'(0) \geq 1$. But any $t \leq 0$ has $\tilde{g}'(t) = 1 + \frac{e^t g'(e^t)}{g(e^t)}$, so the lemma follows if we show $g'(1) \geq 0$.

As we noted in the main text, Assumption 3 amounts to $\theta \mapsto \frac{g(\theta)}{g(\alpha\theta)}\left[\theta - \frac{1-G(\theta)}{g(\theta)}\right]$ being strictly increasing on $(0, 1/\alpha]$ for every $\alpha > 1$. Hence, θ in this range has

$$\begin{aligned} 0 &\leq g(\alpha\theta)^2 \frac{\partial}{\partial\theta}\left\{\frac{g(\theta)}{g(\alpha\theta)}\left[\theta - \frac{1-G(\theta)}{g(\theta)}\right]\right\} \\ &= g(\alpha\theta)\left[g(\theta) + \theta g'(\theta) + g(\theta)\right] - \left\{\theta g(\theta) - [1-G(\theta)]\right\} g'(\alpha\theta)\alpha \\ &= \frac{g(\alpha\theta)g(\theta)}{\theta}\left\{\theta\left[2 + \frac{\theta g'(\theta)}{g(\theta)}\right] - \varphi_{1,1}(\theta)\frac{\alpha\theta g'(\alpha\theta)}{g(\alpha\theta)}\right\}. \end{aligned}$$

Specializing this inequality to the case in which $\theta \in (0,1)$ and $\alpha = 1/\theta$ yields

$$\varphi_{1,1}(\theta)\frac{g'(1)}{g(1)} \leq \theta\left[2 + \frac{\theta g'(\theta)}{g(\theta)}\right] = \theta\left[1 + \tilde{g}'(\log\theta)\right].$$

Because $\int_0^1 \varphi_{1,1} g = 0$ and (by Assumption 3) $\varphi_{1,1}$ is strictly increasing, any sufficiently small $\theta \in (0,1)$ has $\varphi_{1,1}(\theta) < 0$, so that

$$\begin{aligned} \frac{g'(1)}{g(1)} &\geq \limsup_{\theta\searrow 0} \frac{\theta\left[1 + \tilde{g}'(\log\theta)\right]}{\varphi_{1,1}(\theta)} \\ &= \limsup_{\theta\searrow 0} \frac{\theta g(\theta)\left[1 + \tilde{g}'(\log\theta)\right]}{\theta g(\theta) - [1-G(\theta)]} \\ &= \limsup_{t\to-\infty} \frac{e^{\tilde{g}(t)} + e^{\tilde{g}(t)}\tilde{g}'(t)}{e^{\tilde{g}(t)} - [1-G(e^t)]} \\ &= \frac{0+0}{0-(1-0)} = 0. \end{aligned}$$

Therefore, $g'(1) \geq 0$ as required. *Q.E.D.*

Supplementary Appendix

C. On the Linear Demand Environment

Lemma 12. *In the linear demand environment, define* $\Gamma^* : [0, \overline{v}(1)] \to \mathbb{R}$ *via*

$$\Gamma^*(v) := \underline{q}(v) + \frac{v}{\overline{v}'(\underline{q}(v))}$$

for $v \in (0,1]$*, and* $\Gamma^*(0) := 0$*.*

(i) The function Γ^* *is continuous and strictly increasing, and every* $q \in [0,1]$ *has*

$$\overline{v}(q) D_q(\Gamma^*) = \int_0^{\overline{v}(q)} \Gamma^* = q\overline{v}(q).$$

(ii) The function Γ^* *is greedy.*[48] *Conversely, if* $\hat{q} \in [0,1]$ *and* Γ *is greedy up to* $\overline{v}(\hat{q})$*, then* Γ *agrees with* Γ^* *on* $[0, \overline{v}(\hat{q}))$*.*

(iii) A unique $\overline{p}^* \in (0, \overline{v}(1))$ *exists with* $\Gamma^*(\overline{p}^*) = 1$*.*

(iv) Every $\hat{p} \in [0, \overline{p}^*]$ *admits a unique* $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ *such that* $\int_{\hat{p}}^{\overline{v}(\hat{q})} (1 - \Gamma^*) = 0$*, and this* $\hat{q}$ *strictly decreases as* $\hat{p}$ *increases.*

Proof. First, let us observe that Γ^* is continuous and strictly increasing on $(0,1]$. It is continuous there because $\overline{v}$ is continuously differentiable and $\overline{v}'$ is strictly positive there. To see it is strictly increasing (or equivalently, that $\Gamma^* \circ \overline{v}$ is) we apply strict convexity of $h := \frac{1}{\overline{v}}|_{(0,1]}$. If $0 < q_0 < q_1 \leq 1$, strict convexity implies slope $m := \frac{h(q_1) - h(q_0)}{q_1 - q_0}$ has $h'(q_0) < m < h'(q_1) < 0$, and so[49]

$$\begin{aligned}
\Gamma^*(\overline{v}(q_1)) - \Gamma^*(\overline{v}(q_0)) &= q_1 - q_0 - \left[\tfrac{h(q_1)}{h'(q_1)} - \tfrac{h(q_0)}{h'(q_0)}\right] = \tfrac{h(q_0) - h(q_1)}{-m} - \left[\tfrac{h(q_1)}{h'(q_1)} - \tfrac{h(q_0)}{h'(q_0)}\right] \\
&> \tfrac{h(q_0) - h(q_1)}{-h'(q_0)} - \left[\tfrac{h(q_1)}{h'(q_1)} - \tfrac{h(q_0)}{h'(q_0)}\right] = h(q_1)\left[\tfrac{1}{h'(q_0)} - \tfrac{1}{h'(q_1)}\right] > 0.
\end{aligned}$$

[48] Our main text defines greediness only for (increasing and right-continuous) functions $\mathbb{R}_+ \to \mathbb{R}_+$, but the definition can be applied verbatim to a function defined on $[0, \overline{v}(1)]$.

[49] The converse also holds given that $\overline{v}$ is continuously differentiable with $\overline{v}|_{(0,1]} > 0$: strict monotonicity of Γ^* implies strict convexity of h. Indeed, if $0 < q_0 < q_1 \leq 1$ and $h'(q_0) \geq h'(q_1)$, we can replace q_0 with $\min\left(\arg\max_{[q_0,q_1]} h'\right)$ to ensure $m \leq h'(q_0)$. Then, replacing "$>$" with "$\leq$" in the inequality chain yields $\Gamma^*(\overline{v}(q_1)) \leq \Gamma^*(\overline{v}(q_0))$. This equivalence is substantively the same as the observation (McAfee and McMillan, 1987, footnote 11) that a type distribution is regular if and only if the reciprocal of its survival function is convex.

Now, we observe Γ^* is continuous at 0, and so is also globally strictly increasing. And indeed, that $\lim_{q \searrow 0} \overline{v}(q) = 0$ implies $\lim_{q \searrow 0} \log \overline{v}(q) = -\infty$, so that $\limsup_{q \searrow 0} (\log \overline{v})'(q) = \infty$. Therefore,

$$\liminf_{q \searrow 0} \Gamma^*(\overline{v}(q)) = \liminf_{q \searrow 0} \left[q + \tfrac{1}{(\log \overline{v})'(q)} \right] = 0.$$

As $\Gamma^*|_{(0,1]}$ is increasing, it follows that $0 = \lim_{q \searrow 0} \Gamma^*(\overline{v}(q)) = \lim_{v \searrow 0} \Gamma^*(v)$.

Next, given $\hat{q} \in (0, 1]$, let us characterize which increasing and right-continuous functions $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are greedy up to $\overline{v}(\hat{q})$. In light of Lemma 10, we can restrict attention to functions Γ that are continuous on $[0, \overline{v}(\hat{q}))$ with $\Gamma(0) = 0$. Note that such Γ is greedy up to $\overline{v}(\hat{q})$ if and only if $\int_0^{\overline{v}(q)} \Gamma = q\overline{v}(q)$ for every $q \in [0, \hat{q}]$. Because this equality holds for $q = 0$ and both sides are differentiable in q, it holds for every $q \in (0, \hat{q})$ if and only if the derivatives coincide at every $q \in (0, \hat{q})$—that is

$$q\overline{v}'(q) + \overline{v}(q) = \overline{v}'(q)\Gamma\left(\overline{v}(q)\right).$$

Rearranging, Γ is greedy if and only if it agrees with Γ^* on $[0, \overline{v}(\hat{q}))$. The same characterization applies vacuously for $\hat{q} = 0$.

In particular, the equivalence of the previous paragraph tells us Γ^* is greedy. Now, because Γ^* is strictly increasing, at most one $\overline{p}^* \in (0, \overline{v}(1))$ can exist with $\Gamma^*(\overline{p}^*) = 1$. Because Γ^* is continuous and

$$\Gamma^*(0) = 0 < 1 < 1 + \tfrac{\overline{v}(1)}{\overline{v}'(1)} = \Gamma^*(\overline{v}(1)),$$

the intermediate value theorem tells us some such $\overline{p}^*$ exists.

Observe next, because Γ^* is strictly increasing, it follows that the function $[0, \overline{v}(1)] \to \mathbb{R}$ given by $p \mapsto \int_0^p (1 - \Gamma^*)$ is continuous and strictly concave and is maximized at $\overline{p}^*$. Moreover, its value at the right endpoint of its domain is $\int_0^{\overline{v}(1)} (1 - \Gamma^*) = \overline{v}(1) - \int_0^{\overline{v}(1)} \Gamma^* = \overline{v}(1) - 1\overline{v}(1) = 0$, the same as its value at the left endpoint. Therefore, every $\hat{p} \in [0, \overline{p}^*]$ admits a unique $\hat{p}' \in [\overline{p}^*, \overline{v}(1)]$ such that $\int_0^{\hat{p}'} (1 - \Gamma^*) = \int_0^{\hat{p}} (1 - \Gamma^*)$—hence a unique $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ such that $\int_{\hat{p}}^{\overline{v}(\hat{q})} (1-$

$\Gamma^*) = 0$. Moreover, this $\hat{p}'$ continuously strictly decreases as $\hat{p}$ increases, and so too does $\hat{q}$. *Q.E.D.*

In line with the previous lemma, we can introduce the following notations:

Notation 2. *In the linear demand environment:*

(i) Define $\mathcal{Q} : [0, \overline{p}^*] \to [\underline{q}(\overline{p}^*), 1]$ *to be the unique function mapping any* $\hat{p} \in [0, \overline{p}^*]$ *to the unique* $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ *such that* $\int_{\hat{p}}^{\overline{v}(\hat{q})}(1 - \Gamma^*) = 0$.

(ii) Define $\mathcal{P} := \mathcal{Q}^{-1} : [\underline{q}(\overline{p}^*), 1] \to [0, \overline{p}^*]$ *and* $\mathcal{V} := \overline{v} \circ \mathcal{Q} : [0, \overline{p}^*] \to [\overline{p}^*, \overline{v}(1)]$.

(iii) For each $\hat{p} \in [0, \overline{p}^*]$*, define* $\Pi(\cdot|\hat{p}) \in \Delta(\mathbb{R}_+)$ *via*

$$\Pi(p|\hat{p}) := \begin{cases} \Gamma^*(p) & : \ p < \hat{p} \\ 1 & : \ p \geq \hat{p}. \end{cases}$$

(iv) Define $\mathcal{R} : [0, \overline{p}^*] \times [\underline{q}(\overline{p}^*), 1] \to \mathbb{R}$ *by* $\mathcal{R}(\hat{p}, \hat{q}) := R_{\hat{q}}(\Pi(\cdot|\hat{p}))$.

Now, let us record some useful computations about these objects.

Lemma 13. *In the linear demand environment:*

(i) The functions $\mathcal{V}$ *and* $\mathcal{Q}$ *are continuously differentiable on* $[0, \overline{p}^*)$*, and* $\mathcal{P}$ *is continuously differentiable on* $(\underline{q}(\overline{p}^*), 1]$*. Any* $\hat{p} \in [0, \overline{p}^*)$ *and* $\hat{q} = \mathcal{Q}(\hat{p}) \in (\underline{q}(\overline{p}^*), 1]$ *have*

$$\mathcal{V}'(\hat{p}) = -\frac{1 - \Gamma^*(\hat{p})}{\Gamma^*(\overline{v}(\hat{q})) - 1}, \quad \mathcal{Q}'(\hat{p}) = \frac{\mathcal{V}'(\hat{p})}{\overline{v}'(\hat{q})}, \quad \textit{and } \mathcal{P}'(\hat{q}) = \frac{1}{\mathcal{Q}'(\hat{p})},$$

which are all strictly negative.

(ii) Any $\hat{p} \in [0, \overline{p}^*]$ *has*

$$\int_0^\infty p^2 \, \mathrm{d}\Pi(p|\hat{p}) = 2\int_0^{\hat{p}} p[1 - \Gamma^*(p)] \, \mathrm{d}p = \hat{p}^2 \left[1 - \underline{q}(\hat{p})\right] - \int_0^{\underline{q}(\hat{p})} \overline{v}^2.$$

(iii) Any $\hat{p} \in [0, \overline{p}^*]$ *and* $\hat{q} \in [\underline{q}(\overline{p}^*), 1]$ *have*

$$\mathcal{R}(\hat{p}, \hat{q}) = \int_0^{\hat{p}} \left[1 - \tfrac{2p}{\overline{v}(\hat{q})}\right] [1 - \Gamma^*(p)] \, \mathrm{d}p.$$

(iv) Any $\hat{p} \in [0, \overline{p}^*)$ *has* $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right) = \frac{1-\Gamma^*(\hat{p})}{\mathcal{V}(\hat{p})} r(\hat{p})$*, where*

$$r(\hat{p}) := [\mathcal{V}(\hat{p}) - 2\hat{p}] - \frac{2}{\mathcal{V}(\hat{p})\left[\Gamma^*(\mathcal{V}(\hat{p})) - 1\right]} \int_0^{\hat{p}} p\left[1 - \Gamma^*(p)\right]\,\mathrm{d}p.$$

(v) The function $r : [0, \overline{p}^*) \to \mathbb{R}$ *is continuously differentiable with strictly negative derivative.*

(vi) The function $[0, \overline{p}^*] \to \mathbb{R}$ *given by* $\hat{p} \mapsto \mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right)$ *is strictly quasiconcave with interior maximum (where* $r(\cdot)$ *takes value* 0*).*

Proof. We first establish the derivative computations for $\mathcal{V}$, $\mathcal{Q}$, and $\mathcal{P}$. We need only show the given properties for $\mathcal{V}$, and then those for $\mathcal{Q}$ and $\mathcal{P}$ follow directly from the chain rule. At any $\hat{p} \in [0, \overline{p}^*)$, that $\mathcal{Q}(\hat{p}) > \underline{q}(\overline{p}^*)$ implies the partial derivative of the continuously differentiable function $(p, v) \mapsto \int_p^v (1-\Gamma^*)$ with respect to its second argument is nonzero at $(\hat{p}, \mathcal{V}(\hat{p}))$. The implicit function theorem therefore implies $\mathcal{V}$ is differentiable at $\hat{p}$ with

$$0 = \frac{\mathrm{d}}{\mathrm{d}\hat{p}} \int_{\hat{p}}^{\mathcal{V}(\hat{p})} (1 - \Gamma^*) = \mathcal{V}'(\hat{p})\left[1 - \Gamma^*(\mathcal{V}(\hat{p}))\right] - \left[1 - \Gamma^*(\hat{p})\right].$$

Thus, $\mathcal{V}'$ is as desired.

Next, observe that the expectation of the squared price given $\Pi(\cdot|\hat{p})$ is

$$\begin{aligned}
\int_0^\infty p^2\,\mathrm{d}\Pi(p|\hat{p}) &= [1 - \Gamma^*(\hat{p})]\hat{p}^2 + \int_0^{\hat{p}} p^2\,\mathrm{d}\Gamma^*(p) \\
&= \hat{p}^2 - \hat{p}^2\Gamma^*(\hat{p}) + \left[p^2\Gamma^*(p)\right]_{p=0}^{\hat{p}} - \int_0^{\hat{p}} 2p\Gamma^*(p)\,\mathrm{d}p \\
&= \hat{p}^2 - 2\int_0^{\hat{p}} p\Gamma^*(p)\,\mathrm{d}p \\
&= 2\int_0^{\hat{p}} p\left[1 - \Gamma^*(p)\right]\,\mathrm{d}p,
\end{aligned}$$

which is in turn equal to $\hat{p}^2\left[1 - \underline{q}(\hat{p})\right] - \int_0^{\underline{q}(\hat{p})} \overline{v}^2$ because the two expressions are both zero for $\hat{p} = 0$ and have the same derivative with respect to $\hat{p}$.

Toward computing $\mathcal{R}(\hat{p},\hat{q})$, note that

$$\int_0^\infty p\ \mathrm{d}\Pi(p|\hat{p}) = \int_0^\infty [1-\Pi(\cdot|\hat{p})] = \int_0^{\hat{p}} (1-\Gamma^*).$$

Moreover, that $\hat{p} \in [0,\overline{p}^*]$ and $\hat{q} \in [\underline{q}(\overline{p}^*),1]$ implies $\hat{p} \leq \overline{v}(\hat{q})$. Thus,

$$\begin{aligned}
\overline{v}(\hat{q})\mathcal{R}(\hat{p},\hat{q}) &= \overline{v}(\hat{q})\int_0^\infty p\left[1-\tfrac{p}{\overline{v}(\hat{q})}\right]\mathrm{d}\Pi(p|\hat{p}) \\
&= \overline{v}(\hat{q})\int_0^\infty p\ \mathrm{d}\Pi(p|\hat{p}) - \int_0^\infty p^2\ \mathrm{d}\Pi(p|\hat{p}) \\
&= \overline{v}(\hat{q})\int_0^{\hat{p}} (1-\Gamma^*) - 2\int_0^{\hat{p}} p\,[1-\Gamma^*(p)]\,\mathrm{d}p.
\end{aligned}$$

Hence, $\mathcal{R}(\hat{p},\hat{q}) = \int_0^{\hat{p}}\left[1-\frac{2p}{\overline{v}(\hat{q})}\right][1-\Gamma^*(p)]\,\mathrm{d}p$.

Now, because

$$\begin{aligned}
\left.\frac{\partial}{\partial \hat{q}}\right|_{\hat{q}=\mathcal{Q}(\hat{p})}\mathcal{R}(\hat{p},\hat{q}) &= \left\{\int_0^{\hat{p}} 2p\,[1-\Gamma^*(p)]\ \mathrm{d}p\right\}\left.\frac{\partial}{\partial \hat{q}}\right|_{\hat{q}=\mathcal{Q}(\hat{p})}\left[\frac{-1}{\overline{v}(\hat{q})}\right] \\
&= \frac{2\overline{v}'(\hat{q})}{\overline{v}(\hat{q})^2}\int_0^{\hat{p}} p\,[1-\Gamma^*(p)]\ \mathrm{d}p,
\end{aligned}$$

the chain rule yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}(\hat{p},\mathcal{Q}(\hat{p})) &= \frac{\partial}{\partial \hat{p}}\mathcal{R}(\hat{p},\mathcal{Q}(\hat{p})) + \mathcal{Q}'(\hat{p})\left.\frac{\partial}{\partial \hat{q}}\right|_{\hat{q}=\mathcal{Q}(\hat{p})}\mathcal{R}(\hat{p},\hat{q}) \\
&= \left[1-\frac{2\hat{p}}{\overline{v}(\mathcal{Q}(\hat{p}))}\right][1-\Gamma^*(\hat{p})] + \frac{\mathcal{V}'(\hat{p})}{\overline{v}'(\mathcal{Q}(\hat{p}))}\frac{2\overline{v}'(\mathcal{Q}(\hat{p}))}{\overline{v}(\mathcal{Q}(\hat{p}))^2}\int_0^{\hat{p}} p\,[1-\Gamma^*(p)]\ \mathrm{d}p \\
&= \frac{1-\Gamma^*(\hat{p})}{\overline{v}(\mathcal{Q}(\hat{p}))}\left\{[\overline{v}(\mathcal{Q}(\hat{p}))-2\hat{p}] + \frac{2\mathcal{V}'(\hat{p})}{\overline{v}(\mathcal{Q}(\hat{p}))\,[1-\Gamma^*(\hat{p})]}\int_0^{\hat{p}} p\,[1-\Gamma^*(p)]\ \mathrm{d}p\right\} \\
&= \frac{1-\Gamma^*(\hat{p})}{\overline{v}(\mathcal{Q}(\hat{p}))}r(\hat{p}).
\end{aligned}$$

Next, that r is continuously differentiable on $[0,\overline{p}^*)$ follows directly from $\mathcal{V}$ being so and Γ^* being continuous. To see r has strictly negative derivative on $[0,\overline{p}^*)$, it suffices to see that $r(\hat{p})+2\hat{p}$ is decreasing on this range. And indeed, because $\mathcal{V}$ is decreasing there, it follows that $\hat{p} \mapsto r(\hat{p}) + 2\hat{p}$ is a decreasing function minus the ratio of a positive increasing function to a positive decreasing function—and hence is decreasing as desired.

Finally, because $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right)$ is a strictly positive multiple of $r(\hat{p})$, which is strictly decreasing in $\hat{p} \in [0, \overline{p}^*)$, it follows that $\hat{p} \mapsto \mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right)$ is strictly quasiconcave on $[0, \overline{p}^*)$—hence on $[0, \overline{p}^*]$ by continuity. Moreover, $\hat{p} \mapsto \mathcal{R}\left(\hat{p}, \mathcal{Q}(\hat{p})\right)$ is maximized on the interior of its domain if r has an interior root. And indeed, $r(0) = \overline{v}(1) > 0$, whereas any $\hat{p} \in [0, \overline{p}^*)$ close enough to $\overline{p}^*$ has $\mathcal{V}(\hat{p}) < 2\hat{p}$ and so $r(\hat{p}) < 0$. Therefore, r has an interior root by the intermediate value theorem. *Q.E.D.*

Lemma 14. *In the linear demand environment,* (Π^*, q^*) *is optimal if and only if* $\Pi^* = \Pi(\cdot|p^*)$ *and* $q^* = \mathcal{Q}(p^*)$ *for the unique* $p^* \in (0, \overline{p}^*)$ *satisfying* $r(p^*) = 0$.

Proof. First, we observe any optimal (Π^*, q^*) must have $\Pi^* = \Pi(\cdot|\hat{p})$ for some $\hat{p} \in [0, \overline{p}^*]$. To see this, note that Theorem 1 tells us Π^* is greedy up to the top of its support p^*. But then Lemma 12 tells us Π^* agrees with Γ^* on $[0, p^*)$, and so fact that Π^* is in $\Delta(\mathbb{R}_+)$ tells us $\Pi^* = \Pi(\cdot|p^*)$ and $p^* \leq \overline{p}^*$.

Now we argue that, given $\hat{p} \in [0, \overline{p}^*]$, the set of all $q \in (0, 1]$ with $D_q(\Pi(\cdot|\hat{p})) \geq q$ is equal to $[0, \mathcal{Q}(\hat{p})]$. Toward this characterization, first note (given Lemma 12) any $q \in [0, \underline{q}(\hat{p})]$ has $D_q(\Pi(\cdot|\hat{p})) = \int_0^{\overline{v}(q)} \Pi(\cdot|\hat{p}) f_q = \int_0^{\overline{v}(q)} \Gamma^* f_q = D_q(\Gamma^*) = q$. Next observe, any $q \in [\underline{q}(\hat{p}), 1]$ has (again by Lemma 12)

$$\overline{v}(q) D_q(\Pi(\cdot|\hat{p})) = \overline{v}(q) \int_0^{\overline{v}(q)} \Pi(\cdot|\hat{p}) f_q = \int_0^{\overline{v}(q)} \Pi(\cdot|\hat{p}) = \int_0^{\hat{p}} \Gamma^* + \int_{\hat{p}}^{\overline{v}(q)} 1,$$

and so $\overline{v}(q) \left[D_q(\Pi(\cdot|\hat{p})) - q\right] = \int_0^{\hat{p}} \Gamma^* + \int_{\hat{p}}^{\overline{v}(q)} 1 - \int_0^{\overline{v}(q)} \Gamma^* = \int_{\hat{p}}^{\overline{v}(q)} (1 - \Gamma^*)$. Therefore, because Lemma 12 says Γ^* is strictly increasing, the function $q \mapsto \overline{v}(q) \left[D_q(\Pi(\cdot|\hat{p})) - q\right]$ is strictly quasiconcave on $[\underline{q}(\hat{p}), 1]$ and zero at $\underline{q}(\hat{p})$. Because the function also takes value zero at $\mathcal{Q}(\hat{p})$, it is then nonnegative up to $\mathcal{Q}(\hat{p})$ and strictly negative to the right. Thus, $\{q \in (0, 1] : \; D_q(\Pi(\cdot|\hat{p})) \geq q\} = [0, \mathcal{Q}(\hat{p})]$, as desired.

By the previous two paragraphs, we can write the seller's problem (P*) as

$$\max_{\hat{p}\in[0,\overline{p}^*],\ \hat{q}\in[0,1]} R_{\hat{q}}(\Pi(\cdot|\hat{p})) \text{ s.t. } \hat{q} \le \mathcal{Q}(\hat{p}).$$

Because Lemma 4 tells us the objective is strictly increasing (wherever strictly positive, as the optimal revenue is) in the quantity, the seller optimally sets $\hat{q} = \mathcal{Q}(\hat{p})$, and so her problem can be written as

$$\max_{\hat{p}\in[0,\overline{p}^*]} \mathcal{R}(\hat{p}, \mathcal{Q}(\hat{p})).$$

By Lemma 13, this objective is strictly quasiconcave with interior optimum, and the optimum p^* is characterized by $r(p^*) = 0$. *Q.E.D.*

Lemma 15. *Consider the linear demand environment with $\overline{v}(q) = q$ for every $q \in [0,1]$, let (Π^*, q^*) be optimal, and let p^* be the maximum of the support of Π^*. Then $p^* \in \left(0, \frac{1}{2}\right)$ is the unique zero of the strictly decreasing function $r : (0, \frac{1}{2}) \to \mathbb{R}$ given by*

$$r(\hat{p}) = \tfrac{1}{3(1-\hat{p})(1-2\hat{p})}\left[3(1-\hat{p})(1-2\hat{p})(1-3\hat{p}) - \hat{p}^2(3-4\hat{p})\right],$$

and the induced consumer surplus is equal to $CS_{q^*}(\Pi^*) = \frac{(1-p^*)^2}{3} + \frac{(1-2p^*)^3}{6(1-p^*)}$.

In particular, $p^* \approx 0.28$ has $p^* < \frac{1}{2}$, and $CS_{q^*}(\Pi^*) \approx 0.19$ has $CS_{q^*}(\Pi^*) > \frac{1}{6}$.

Proof. Lemma 14 tells us the unique optimal (Π^*, q^*) is given by $\Pi^*(p) = \Gamma^*(p)\mathbf{1}_{p<p^*} + \mathbf{1}_{p\ge p^*} = 2p\mathbf{1}_{p<p^*} + \mathbf{1}_{p\ge p^*}$ and $q^* = 1 - p^*$, where (by Lemma 13) $p^* \in \left(0, \frac{1}{2}\right)$ is the unique zero of the decreasing function r given by

$$\begin{aligned} r(\hat{p}) &= [(1-\hat{p}) - 2\hat{p}] - \frac{2}{(1-\hat{p})\left[2(1-\hat{p})-1\right]}\int_0^{\hat{p}} p(1-2p)\,\mathrm{d}p \\ &= 1 - 3\hat{p} - \frac{2}{(1-\hat{p})(1-2\hat{p})}\left(\tfrac{1}{2}\hat{p}^2 - \tfrac{2}{3}\hat{p}^3\right) \\ &= \tfrac{1}{3(1-\hat{p})(1-2\hat{p})}\left[3(1-\hat{p})(1-2\hat{p})(1-3\hat{p}) - \hat{p}^2(3-4\hat{p})\right]. \end{aligned}$$

The consumer surplus is then given by

$$\begin{aligned}
\mathrm{CS}_{q^*}(\Pi^*) &= \int_{\mathbb{R}_+} \int_0^{q^*} (v-p)_+ \left(\tfrac{1}{q^*}\right) \, \mathrm{d}v \, \mathrm{d}\Pi^*(p) \\
&= \tfrac{1}{q^*} \int_{[0,p^*]} \int_p^{q^*} (v-p) \, \mathrm{d}v \, \mathrm{d}\Pi^*(p) \\
&= \tfrac{1}{2q^*} \int_{[0,p^*]} (q^*-p)^2 \, \mathrm{d}\Pi^*(p) \\
&= \tfrac{1}{2q^*} \left[\int_0^{p^*} (q^*-p)^2 \, \mathrm{d}(2p) + (1-2p^*)(q^*-p^*)^2 \right] \\
&= \tfrac{1}{6q^*} \left\{ 2\left[(q^*)^3 - (q^*-p^*)^3 \right] + 3(1-2p^*)^3 \right\} \\
&= \frac{(1-p^*)^2}{3} + \frac{(1-2p^*)^3}{6(1-p^*)}.
\end{aligned}$$

Finally, the last sentence follows from a numerical computation. *Q.E.D.*

D. Proofs for Section 4

D.1. Inputs for the proof of Proposition 2

We begin by describing the complete-information benchmark more formally. In the complete-information specification of the game, we assume that the seller makes price offers $(p_{i,\theta})_{(i,\theta)\in I\times\Theta}$, and can choose an arbitrary measurable pricing policy $I \times \Theta \to \mathbb{R}_+$. Just as the seller's choice amounts to a price distribution in our main model, here it amounts to a contingent price distribution (**CPD**). Given a contingent price distribution $\mathbf{\Pi}$, let $\mathbf{\Pi}(\cdot|\theta)$ denote both the conditional price distribution and its CDF for each type $\theta \in \Theta$, and let Π denote the induced marginal price distribution. Hence, recalling that $G \in \Delta(\Theta)$ denotes the (continuous) distribution of types, we have $\Pi(p) = \int_\Theta \mathbf{\Pi}(p|\cdot) \, \mathrm{d}G$ for every price $p \geq 0$.

As in our main model, we assume buyers purchase when indifferent. Given a contingent price distribution $\mathbf{\Pi}$, the total quantity demanded for anticipated quantity q is

$$D_q(\mathbf{\Pi}) = \int_\Theta \mathbf{\Pi}(u(\theta, q)|\theta) \ \mathrm{d}G(\theta),$$

and the induced revenue is

$$R_q(\mathbf{\Pi}) = \int_\Theta \int_{[0,u(\theta,q)]} p \ \mathrm{d}\mathbf{\Pi}(p|\theta) \ \mathrm{d}G(\theta).$$

As in our model, we assume the seller evaluates a contingent price distribution according to its worst-case equilibrium quantity (which exists, as our analysis establishes).

We define a topology on the space of contingent price distributions as follows. Say a net $(\mathbf{\Pi}_k)_k$ of contingent price distributions converges to one $\mathbf{\Pi}$ if their induced joint distributions of types and prices (weakly) converges—formally, $\int_\Theta \int_{\mathbb{R}_+} \psi(\theta, p) \ \mathrm{d}\mathbf{\Pi}_k(p|\theta) \ \mathrm{d}G(\theta) \to \int_\Theta \int_{\mathbb{R}_+} \psi(\theta, p) \ \mathrm{d}\mathbf{\Pi}(p|\theta) \ \mathrm{d}G(\theta)$ for every bounded continuous $\psi : \Theta \times \mathbb{R}_+ \to \mathbb{R}$. Then, just as in our main model, we say that $(\mathbf{\Pi}, q)$ is **complete-information optimal** if it is a limit point of a sequence $(\mathbf{\Pi}_k, q_k)_k$ such that quantity q_k is the worst-case equilibrium quantity given price distribution $\mathbf{\Pi}_k$ for every k and $R_{q_k}(\mathbf{\Pi}_k)$ converges to the seller's complete-information optimal value

$$\sup_{\mathbf{\Pi}:\Theta\to\Delta(\mathbb{R}_+) \text{ measurable}} \ \min_{q\in[0,1]} \ R_q(\mathbf{\Pi})$$
$$\text{subject to } \ D_q(\mathbf{\Pi}) = q.$$

We begin by noting that Lemma 2 and Lemma 3 generalize to the case of a contingent price distribution (as does their supporting Lemma 1) with the proofs applying mutatis mutandis. When invoking these lemmas in what follows, we mean the appropriate generalizations.

Notation 3. *Let* $\mathcal{N}$ *denote the set of positive measures* η *on* $\Theta \times [0, 1]$ *whose first marginal* $\mathrm{marg}_1\eta$ *is a submeasure of the type distribution* G.

Lemma 16. *Any contingent price distribution* $\mathbf{\Pi}$ *induces some measure* $\eta_{\mathbf{\Pi}} \in \mathcal{N}$ *by letting* $\eta_{\mathbf{\Pi}}\left(\hat{\Theta} \times [0, \hat{q}]\right) := \int_{\hat{\Theta}} \mathbf{\Pi}\left(u(\theta, \hat{q})|\theta\right) \mathrm{d}G(\theta)$ *for every* $\hat{q} \in [0, 1]$ *and* *measurable* $\hat{\Theta} \subseteq \Theta$*; and conversely every element of* $\mathcal{N}$ *is equal to* $\eta_{\mathbf{\Pi}}$ *for some contingent price distribution* $\mathbf{\Pi}$*. Moreover, any contingent price distribution* $\mathbf{\Pi}$ *and anticipated quantity* $q \in [0,1]$ *generate demand and revenue*

$$D_q(\mathbf{\Pi}) = \text{marg}_2\ \eta_{\mathbf{\Pi}}[0,q] \text{ and } R_q(\mathbf{\Pi}) = \int_{\Theta\times[0,q]} u\ \mathrm{d}\eta_{\mathbf{\Pi}}.$$

Proof. Define the measurable function $\varphi : \Theta \times \mathbb{R}_+ \to \Theta \times ([0,1] \cup \{\infty\})$ by letting $\varphi(\theta,p) := (\theta,\ u(\theta,\cdot)^{-1}(p))$ if $p \leq u(\theta,1)$, and $\varphi(\theta,p) := (\theta,\infty)$ otherwise. Given a contingent price distribution $\mathbf{\Pi}$, consider the pushforward by φ of the $(G,\mathbf{\Pi})$ joint distribution of $\Theta\times\mathbb{R}_+$, and restrict that measure to $\Theta\times[0,1] \subseteq \Theta \times ([0,1]\cup\{\infty\})$. Because φ fixes the first coordinate, this measure has a submeasure of G as its first marginal. Because the conditions defining $\eta_{\mathbf{\Pi}}$ apply to this measure, and the events $\{\hat{\Theta}\times[0,\hat{q}] :\ \hat{\Theta} \subseteq \Theta \text{ measurable},\ 0 \leq \hat{q} \leq 1\}$ generate the full Borel field of $\Theta \times [0,1]$, no other measure satisfies these equations. The first assertion follows.

Toward the converse in the first sentence, take any $\eta \in \mathcal{N}$, and let $\eta_1 := \text{marg}_1\eta$. Because η_1 is a submeasure of G, the Radon-Nikodym theorem delivers a measurable function $h : \Theta \to [0,1]$ such that $\eta_1(\hat{\Theta}) = \int_{\hat{\Theta}} h\ \mathrm{d}G$ for every measurable $\hat{\Theta} \subseteq \Theta$. The disintegration theorem (Theorem 3.4, Kallenberg, 1997) delivers measurable $\eta^2 : \Theta \to \Delta[0,1]$ such that $\eta(\hat{\Theta}\times\hat{Q}) = \int_{\hat{\Theta}} \eta^2(\hat{Q}|\cdot)\ \mathrm{d}\eta_1$ for every measurable $\hat{\Theta} \subseteq \Theta$ and $\hat{Q} \subseteq [0,1]$. The contingent price distribution $\mathbf{\Pi}$ given by $\mathbf{\Pi}(p|\theta) := h(\theta)\ \eta^2 \circ u(\theta,\cdot)^{-1}\,([0,p]) + [1-h(\theta)]\ \mathbf{1}_{p\geq\overline{v}(1)+42}$ then has $\eta_{\mathbf{\Pi}} = \eta$.

The formula for $D_q(\mathbf{\Pi})$ is exactly its definition, and a (type-dependent) change of variables $\hat{q} \mapsto u(\theta, \hat{q}) \equiv p$ yields

$$\begin{aligned} R_q(\mathbf{\Pi}) &= \int_\Theta \left[\int_{[0,u(\theta,q)]} p \, \mathrm{d}\mathbf{\Pi}(p|\theta) \right] \, \mathrm{d}G(\theta) \\ &= \int_\Theta \left[\int_0^q u(\theta, \hat{q}) \, \mathrm{d} \left[\mathbf{\Pi}(u(\theta, \cdot) \mid \theta) \right] (\hat{q}) \right] \, \mathrm{d}G(\theta) \\ &= \int_\Theta \int_0^q u(\theta, \hat{q}) \, \mathrm{d} \left[\mathbf{\Pi}(\varphi^{-1}(\cdot) \mid \theta) \right] (\hat{q}) \, \mathrm{d}G(\theta) \\ &= \int_{\Theta \times [0,q]} u \, \mathrm{d}\eta_{\mathbf{\Pi}}, \end{aligned}$$

as desired. *Q.E.D.*

Lemma 17. *A pair* $(\eta, q) \in \mathcal{N} \times [0,1]$ *is optimal in the program*

$$\max_{\eta \in \mathcal{N},\ q \in [0,1]} \int_{\Theta \times [0,q]} u \, \mathrm{d}\eta \qquad (\mathrm{P}^C_{\mathcal{N}})$$

$$\textit{subject to} \ \ \mathrm{marg}_2 \eta[0, \hat{q}] \geq \hat{q} \ \ \forall \hat{q} \in (0, q),$$

if and only if $q = 1$ *and* η *solves the optimal transport problem*

$$\max_{\eta \in \Delta(\Theta \times [0,1])} \int u \, \mathrm{d}\eta \qquad (\mathrm{P}^T)$$

$$\textit{subject to} \ \ \mathrm{marg}_1 \eta = G, \ \mathrm{marg}_2 \eta \ \textit{uniform},$$

and these programs admit optimal solutions.

Moreover, if u *is strictly supermodular, then the unique optimum* η *in program* (P^T) *is supported on* $\{(\theta, G(\theta)) : \ \theta \in \Theta\}$.

Proof. Before studying program ($\mathrm{P}^C_{\mathcal{N}}$), recall some assumed features of u. Because $u(\cdot, \cdot) \geq 0$ is G-a.e. strictly increasing, it follows that every $q \in (0,1]$ has $u(\cdot, q)$ is G-a.e. strictly positive. Meanwhile, the maximum of the supports of $G \circ u(\cdot, 0)^{-1}$ and $G \circ u(\cdot, 1)^{-1}$ are $\overline{v}(0) = 0$ and $\overline{v}(1)$, respectively. We may therefore assume without loss that u is $[0, \overline{v}(1)]$-valued, and in particular bounded.

We turn now to program ($\mathrm{P}^C_{\mathcal{N}}$). For any $\eta \in \mathcal{N}$, let

$$q_\eta := \max \{q \in [0,1] : \ \mathrm{marg}_2\eta[0,\hat{q}] \geq \hat{q} \ \ \forall \hat{q} \in (0,q)\},$$

which exists because any sequence $\{q_k\}_k \subseteq [0,1]$ has $(0, \sup_k q_k) = \bigcup_k (0, q_k)$. Mildly abuse notation by letting $R_q(\eta) := \int_{\Theta \times [0,q]} u \ \mathrm{d}\eta$ for any $\eta \in \mathcal{N}$ and $q \in [0,1]$.

First, observe that any feasible pair (η, q) with $q = 0$ can be strictly improved to one with $q > 0$. Indeed, if $q = 0$, then a.e.-monotonicity of u and the assumption that $\overline{v}(0) = 0$ implies $R_q(\eta) = 0$. But then the feasible pair

$$\left(G \otimes \left[\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_{\frac{1}{2}}\right], \ 1\right)$$

yields strictly higher value $\frac{1}{2} \int u\left(\cdot, \frac{1}{2}\right) \ \mathrm{d}G > 0$.

Next, let us see that any feasible pair (η, q) with $q_\eta \in (0,1)$ can be strictly improved to one of the form $(\tilde{\eta}, 1)$ for $\tilde{\eta} \in \mathcal{N}$. Indeed, take any such (η, q), and let $\eta_2 := \mathrm{marg}_2\eta$. Given the previous paragraph, we may assume without loss that $q > 0$. That (η, q) is feasible and $q_\eta < 1$ implies $\eta_2[0,q] < 1$. So letting $\varepsilon := 1 - \eta_2[0,q] \in (0,1]$, some $\hat{\eta} \in \Delta(\Theta \times [0,q])$ exists such that η is the sum of $(1-\varepsilon)\hat{\eta}$ and some positive measure on $\Theta \times (q,1]$. Letting $\gamma := \frac{1}{\varepsilon}[G - (1-\varepsilon)\mathrm{marg}_1\hat{\eta}]$, define $\tilde{\eta} := (1-\varepsilon)\hat{\eta} + \varepsilon(\gamma \otimes \delta_q)$. By construction, $\tilde{\eta}$ is a probability measure on $\Theta \times [0,q]$ with first marginal G, and so is in $\mathcal{N}$. Moreover, every $\hat{q} \in (0,q)$ has $\mathrm{marg}_2\tilde{\eta}[0,\hat{q}] = \eta_2[0,\hat{q}] \geq \hat{q}$ and every $\hat{q} \in [q,1]$ has $\mathrm{marg}_2\tilde{\eta}[0,\hat{q}] = 1 \geq \hat{q}$, so that $(\tilde{\eta}, 1)$ is feasible. This modification is a strict improvement because $R_1(\tilde{\eta}) - R_q(\eta) = \varepsilon \int u(\cdot, q) \ \mathrm{d}\gamma > 0$.

Now, we show that any feasible pair (η, q) with $\eta_2 := \mathrm{marg}_2\eta$ not equal to the uniform measure on $[0,1]$ can be strictly improved to some $(\tilde{\eta}, \tilde{q})$ that does enjoy this property. Given the previous two paragraphs, we may assume without loss that $q_\eta = 1$. Hence, because u is nonnegative (and so $q \mapsto R_q(\eta)$ increasing), we may assume without loss that $q = 1$. That $q_\eta = 1$ tells us that η is a probability measure and $\eta_2[0,\hat{q}] \geq \hat{q}$ for every $\hat{q} \in [0,1]$—that is $\eta_2 \preceq_{\mathrm{FOSD}} \mathcal{U}$, where $\mathcal{U} \in \Delta[0,1]$ is the uniform measure. Therefore, some increasing $\psi : [0,1] \to [0,1]$ exists such that $\eta_2 = \mathcal{U} \circ \psi^{-1}$ and $\psi(z) \leq z$ for every $z \in [0,1]$. Moreover, that $\eta_2 \neq \mathcal{U}$ tells us $\mathcal{U}\{z \in [0,1]: \psi(z) < z\}$ is strictly positive. Using ψ, let us construct an improvement. To that end, note that the disintegration theorem (Theorem 3.4, Kallenberg, 1997) delivers a measurable function $\eta^1 : [0,1] \to \Delta\Theta$ such that $\eta(\hat{\Theta} \times \hat{Q}) = \int_{\hat{Q}} \eta^1(\hat{\Theta}|\cdot)\, \mathrm{d}\eta_2$ for every measurable $\hat{\Theta} \subseteq \Theta$ and $\hat{Q} \subseteq [0,1]$. So define $\tilde{\eta} \in \Delta(\Theta \times [0,1])$ by letting $\tilde{\eta}(\hat{\Theta} \times \hat{Z}) = \int_{\hat{Z}} \eta^1(\hat{\Theta}|\psi(z))\, \mathrm{d}z$ for every measurable $\hat{\Theta} \subseteq \Theta$ and $\hat{Z} \subseteq [0,1]$. Because $\eta_2 = \mathcal{U} \circ \psi^{-1}$, it follows by construction that $\mathrm{marg}_1\tilde{\eta} = \mathrm{marg}_1\eta = G$, so that $\tilde{\eta} \in \mathcal{N}$. Moreover, we clearly have $\mathrm{marg}_2\tilde{\eta} = \mathcal{U}$ by construction. Finally,

$$R_1(\tilde{\eta}) - R_1(\eta) = \int [u(\theta, z) - u(\theta, \psi(z))]\ \mathrm{d}\tilde{\eta}(\theta, z),$$

which is strictly positive because u is a.e. strictly increasing in its second argument, $\psi(z) \leq z$ for all $z \in Z$, and $\mathcal{U}\{z \in [0,1]: \psi(z) < z\} > 0$.

Next, note if $\eta \in \mathcal{N}$ has $\mathrm{marg}_2\eta$ uniform, then $(\eta, 1)$ is a feasible pair and generates strictly higher value than (η, q) does for any $q \in [0,1)$. Feasibility is immediate, and the revenue ranking comes from noticing that $R_1(\eta) - R_q(\eta) = \int_{\Theta \times (q,1]} u$, which is strictly positive because the integrand is strictly positive and the integrating (positive) measure has mass $1 - q > 0$.

Composing the above arguments tells us: any pair of the form $(\eta, 1)$ with $\mathrm{marg}_2\eta$ uniform is feasible in the program ($\mathrm{P}^C_{\mathcal{N}}$), and any feasible pair not of this form can be strictly improved to one of this form. Therefore, a pair (η, q) is optimal in the program if and only if $q = 1$ and η is an optimal solution to program (P^T). Existence then follows because the domain of program (P^T) is weak* compact and the objective continuous. Finally, the last assertion follows directly from Theorem 4.7 in Galichon (2018). *Q.E.D.*

Lemma 18. *Given a contingent price distribution* $\mathbf{\Pi}$ *and a quantity* $q \in [0,1]$, *the pair* $(\mathbf{\Pi}, q)$ *is complete-information optimal if and only if* $q = 1$ *and* $\eta_{\mathbf{\Pi}}$ *(as defined in the statement Lemma 16) is optimal in program* (P^T) *(as defined in the statement of Lemma 17). Moreover, such an optimum exists.*

Proof. Consider the modification of program (P^*) that replaces $\Delta(\mathbb{R}_+)$ with the space CPD of all contingent price distributions:

$$\max_{\mathbf{\Pi}\in\text{CPD},\ q\in[0,1]} R_q(\mathbf{\Pi}) \qquad (\text{P}^C)$$
$$\text{subject to}\ \ D_{\hat{q}}(\mathbf{\Pi}) \geq \hat{q} \ \ \forall \hat{q} \in (0,q).$$

Observe, Lemma 16 implies $(\mathbf{\Pi}, q)$ is an optimal solution to program (P^C) if and only if $(\eta_{\mathbf{\Pi}}, q)$ is an optimal solution to program ($\text{P}^C_{\mathcal{N}}$); and Lemma 17 says the latter holds if and only if $q = 1$ and $\eta_{\mathbf{\Pi}}$ is an optimal solution to program (P^T), and that such an optimum exists.

The lemma will then follow if we establish that the complete-information optimal pairs are exactly those that are optimal in program (P^C). This equivalence follows from adapting the proof of Proposition 1. In that proof, all of items (ii),(i),(iii)—and the argument that these items and items (iv) and (v) collectively imply the equivalence with program (P^*)—apply mutatis mutandis to the case of *contingent* price distributions.[50] The only parts of the proof that are special to the incomplete-information case are items (iv) and (v). But we have already argued item (iv) in the first paragraph of the present proof, and item (v) holds vacuously for program (P^C), because (as we have argued) every optimal pair $(\mathbf{\Pi}, q)$ for this program has $q = 1$. The lemma follows. *Q.E.D.*

D.2. Proof of Proposition 2

Lemma 18 tells us a complete-information optimum $(q^C, \mathbf{\Pi}^C)$ exists and every one has $q^C = 1$ and $\text{marg}_2\ \eta_{\mathbf{\Pi}^C}[0,q] = q$ for every $q \in [0,1]$. Given the latter condition, Lemma 16 implies $\mathbf{\Pi}^C$ is greedy, hence greedy up to $\overline{v}(1)$. Moreover, because $\eta_{\mathbf{\Pi}^C} \in \Delta(\Theta \times [0,1])$, it follows from the definition of $\eta_{\mathbf{\Pi}^C}$ that $\mathbf{\Pi}^C(u(\theta,1) \mid \theta) = 1$ for G-almost every type θ. Because $u(\theta,1) \leq \overline{v}(1)$ for G-almost every type θ, it follows that $\Pi^C(\overline{v}(1)) = 1$.

Having established the parts of the proposition that apply to the general case, we now turn to linear demand. Because the function $(\theta, q) \mapsto \theta\overline{v}(q)$

[50] That proof invokes Lemma 2 and Lemma 3. As we note in Remark 5, those results also extend without change to the domain of contingent price distributions.

is strictly supermodular, Lemma 17 implies the essentially uniquely optimal contingent price distribution offers each type $\theta \in \Theta$ (a degenerate distribution on) price $u(\theta, \theta) = \theta\overline{v}(\theta)$. Because this function is increasing in θ, the highest supported price is then $p^C = 1\overline{v}(1) > \overline{v}(q^*) > p^*$, where the last two inequalities are given by Theorem 1. For the low-price ranking, recall that Theorem 1 and Lemma 12 tell us $p^* > 0$ and every $q \in [0, \underline{q}(p^*))$ have $\int_0^{\overline{v}(q)} \Pi^* = q\overline{v}(q)$, whereas every $\theta \in [0,1]$ has $\Pi^C(\theta\overline{v}(\theta)) = \theta$. Now suppose $k > 0$ and $\alpha > 1$ exist such that $\frac{1}{\alpha}q^k \le \overline{v}(q) \le \alpha q^k$ for small enough $q \in (0,1)$. Then, for any small enough $p > 0$, type $\theta_p = (\frac{p}{\alpha})^{\frac{1}{k+1}}$ has $\theta_p\overline{v}(\theta_p) \le p$, and so $\Pi^C(p) \ge \theta_p$. So small $q > 0$ has

$$\frac{\int_0^{\overline{v}(q)} \Pi^*}{\int_0^{\overline{v}(q)} \Pi^C} = \frac{q\overline{v}(q)}{\int_0^{\overline{v}(q)} \Pi^C} \le \frac{\alpha q^{k+1}}{\int_0^{\frac{1}{\alpha}q^k} (\frac{p}{\alpha})^{\frac{1}{k+1}} \, \mathrm{d}p} \propto \frac{q^{k+1}}{(q^k)^{\frac{1}{k+1}+1}} = q^{\frac{1}{k+1}} \xrightarrow{q \searrow 0} 0.$$

L'Hôpital's rule then yields $0 = \lim_{p \searrow 0} \frac{\int_0^p \Pi^*}{\int_0^p \Pi^C} = \lim_{p \searrow 0} \frac{\Pi^*(p)}{\Pi^C(p)}$, which in particular implies $\Pi^*(p) < \Pi^C(p)$ for all sufficiently small $p > 0$. Applying this conclusion to the case of $k = 1$ delivers the low-price ranking when $\overline{v}'(0) > 0$.[51]

All that remains now is to show the consumer surplus ranking is ambiguous, even with linear demand. Under complete information, because each type θ is offered a price of $\theta\overline{v}(\theta)$ and buys, that type's surplus is equal to $\theta\overline{v}(1) - \theta\overline{v}(\theta)$; hence the complete-information consumer surplus is equal to $\int_0^1 \theta \left[\overline{v}(1) - \overline{v}(\theta)\right] \, \mathrm{d}\theta$. We will show this quantity can be higher or lower than consumer surplus under incomplete information, depending on parameter $\overline{v}$.

Begin with the leading example in which $\overline{v}(q) = q$ for every $q \in [0,1]$. For this example, Lemma 15 tells us that under incomplete information, the highest supported price has $p^* \approx 0.28$ and consumer surplus $\mathrm{CS}^* := \mathrm{CS}_{q^*}(\Pi^*)$ has $\mathrm{CS}^* \approx 0.19 > \frac{1}{6}$. Under complete information, the consumer surplus for

[51] Even if $\overline{v}'(0) = 0$, this reasoning applies to $\overline{v}$ that is analytic at 0, letting $k \in \mathbb{N}$ be the lowest order derivative that is nonzero (and hence strictly positive because $\overline{v} \ge 0$).

this example is

$$\mathrm{CS}^C = \int_0^1 \theta(1-\theta)\ \mathrm{d}\theta = \tfrac{1}{2}1^2 - \tfrac{1}{3}1^3 = \tfrac{1}{6}.$$

Hence, $\mathrm{CS}^C < \mathrm{CS}^*$, as required.

We now construct an example in which the consumer surplus is strictly higher with complete information. To do so, letting $q^* \approx 0.72$ and $p^* \approx 0.28$ be the optimal quantity and highest supported price under the specification studied in the previous paragraph, we will discuss instances of our model that still have $\overline{v}(q) = q$ for every $q \in [0, q^*]$. Such models will still have $\mathcal{V}(p^*) = q^*$ and will have $\Gamma^*|_{[0,q^*]}$ be the same as in the linear-$\overline{v}$ example. Hence, by Lemma 14, the same pair (Π^*, q^*) remains optimal in this alternate specification. Moreover, because we are leaving $\Gamma^*|_{[0,q^*]}$ (and hence Π^*) fixed, the consumer surplus under incomplete information will be the same CS^* computed above. So our goal is to modify $\overline{v}|_{(q^*,1]}$ in order to raise the complete-information consumer surplus above CS^*.[52] To that end, for each $\varepsilon \geq 0$, define the function $\overline{v}_\varepsilon : [0,1] \to \mathbb{R}$ by letting

$$\overline{v}_\varepsilon(q) := \begin{cases} q & : \ q \in [0, q^*] \\ \dfrac{(q^*)^2}{2q^* - q + \varepsilon(q - q^*)^2} & : \ q \in [q^*, 1]. \end{cases}$$

This function is well-defined—with strictly positive denominator, and with the two definitions at q^* agreeing. Moreover, it is smooth on $[0, q^*]$ and $[q^*, 1]$, and the two one-sided derivatives at q^* agree, making it continuously differentiable. Further, the function $\frac{1}{\overline{v}}$ is strictly convex on $(0, q^*]$ and, if $\varepsilon > 0$, on $[q^*, 1]$. Finally, when $\varepsilon > 0$ is small enough, we have $\overline{v}_\varepsilon' > 0$ globally, making $\overline{v}_\varepsilon$ a valid specification of the linear model for $\varepsilon > 0$ sufficiently small. For any $\varepsilon \geq 0$, let $\mathrm{CS}^C_\varepsilon := \int_0^1 \theta \left[\overline{v}_\varepsilon(1) - \overline{v}_\varepsilon(\theta)\right]\ \mathrm{d}\theta$. Observe $\overline{v}_\varepsilon \to \overline{v}_0$ uniformly as $\varepsilon \searrow 0$, and so $\mathrm{CS}^C_\varepsilon \to \mathrm{CS}^C_0$ too. It therefore suffices to see that $\mathrm{CS}^C_0 > \mathrm{CS}^*$. To that end,

[52] Although this feature is not relevant to establishing the proposition, the $\overline{v}$ we construct make consumer surplus as high as possible subject to preserving $\overline{v}(q) = q$ for $q \in [0, q^*]$.

note that $\overline{v}_0(1) = \frac{(q^*)^2}{2q^*-1} = \frac{(1-p^*)^2}{1-2p^*}$, and so

$$\begin{aligned}
\mathrm{CS}_0^C &= \int_0^1 \theta\left[\overline{v}_0(1) - \overline{v}_0(\theta)\right]\,\mathrm{d}\theta \\
&= \tfrac{1}{2}\overline{v}_0(1) - \int_0^{q^*} \theta^2\,\mathrm{d}\theta - \int_{q^*}^1 \theta\tfrac{(q^*)^2}{2q^*-\theta}\,\mathrm{d}\theta \\
&= \tfrac{(1-p^*)^2}{2(1-2p^*)} - \tfrac{1}{3}\left(q^*\right)^3 + \left(q^*\right)^2\Big[\theta + 2q^*\log(2q^*-\theta)\Big]_{\theta=q^*}^1 \\
&= \tfrac{(1-p^*)^2}{2(1-2p^*)} - \tfrac{1}{3}\left(1-p^*\right)^3 + \left(1-p^*\right)^2\left[1 - q^* + 2q^*\log\tfrac{2q^*-1}{q^*}\right] \\
&= \left(1-p^*\right)^2\left[\tfrac{1}{2(1-2p^*)} + \tfrac{1}{3}\left(4p^*-1\right) + 2\left(1-p^*\right)\log\tfrac{1-2p^*}{1-p^*}\right].
\end{aligned}$$

A computation shows $\mathrm{CS}_0^C \approx 0.24$, and so $\mathrm{CS}_0^C > \mathrm{CS}^*$ as required. *Q.E.D.*

D.3. Inputs for the proof of Proposition 3

Lemma 19. *Ranging over all specifications of the linear demand model, the ratio $\overline{v}(q^*)/\overline{v}(1)$ can be arbitrarily close to zero.*

Proof. Given some $\gamma \in (0, \frac{1}{3})$, let $q := 1-3\gamma^2 \in (0,1)$ and $\hat{q} := 1-\gamma^2 \in (q,1)$, and let $h : (0,1] \to \mathbb{R}$ be the unique function with $h|_{(0,\hat{q}]}$ and $h|_{[\hat{q},1]}$ affine and

$$\left(h(1),\ h(\hat{q}),\ h(q)\right) = \left(1,\ \tfrac{1+\gamma}{\gamma},\ \tfrac{1-q}{1-\hat{q}}h(\hat{q})\right).$$

In what follows, we will show there is a sequence $(\overline{v}_k)_k$ of linear-demand models such that $h_k := 1/\overline{v}_k|_{(0,1]}$ have

$$\left(h_k(q),\ h_k(\hat{q}),\ h_k^+(\hat{q})\right) = \left(h(q),\ h(\hat{q}),\ h'(\hat{q})\right),$$

and the sequence $(h_k)_k$ converges pointwise to h and is uniformly bounded below by some positive constant; and we will show (appending a k subscript to any previously defined object when applied to model $\overline{v}_k$) that for any such sequence any large enough k has $r_k(\mathcal{P}_k(\hat{q})) > 0$. Before doing so, let us see how the lemma would follow. Indeed, first note that for any k, the identity $\frac{1-q}{1-\hat{q}}h_k(\hat{q}) = h_k(q)$ rearranges to $(1-q)\overline{v}_k(q) = (1-\hat{q})\overline{v}_k(\hat{q})$, and so Lemma 12 tells us $\overline{v}_k(q) = \mathcal{P}_k(\hat{q})$. But then, large enough k have $r_k(\mathcal{P}_k(\hat{q})) > 0 = r_k(\mathcal{P}_k(q_k^*))$ by Lemma 14, and $r_k \circ \mathcal{P}_k$ is increasing by Lemma 13. Therefore, such k have $q_k^* < \hat{q}$, and so

$$\frac{\overline{v}_k(q_k^*)}{\overline{v}_k(1)} < \frac{\overline{v}_k(\hat{q})}{\overline{v}_k(1)} \xrightarrow{k\to\infty} \frac{h(1)}{h(\hat{q})} = \frac{\gamma}{1+\gamma}.$$

Because $\frac{\gamma}{1+\gamma}$ can in turn be made arbitrarily small by making $\gamma \in (0, \frac{1}{3})$ small, the lemma would then follow. So we now turn to verifying these two remaining claims.

Before confirming the above two missing pieces, let us compute the one-sided derivatives of h at $\hat{q}$, which which are useful for both. The right and left derivative, respectively, are given by

$$\begin{aligned} h^+(\hat{q}) &= \frac{h(1) - h(\hat{q})}{1-\hat{q}} = \frac{-1}{1-\hat{q}}\,\frac{1}{\gamma}, \\ h^-(\hat{q}) &= \frac{h(\hat{q}) - h(q)}{\hat{q}-q} = \frac{1 - \frac{1-q}{1-\hat{q}}}{\hat{q}-q}\, h(\hat{q}) = \frac{-1}{1-\hat{q}} h(\hat{q}). \end{aligned}$$

Now, let us show the hypotheses on $(h_k)_k$ ensure large k have $r_k(\mathcal{P}_k(\hat{q}))$. To that end, observe:

$$\begin{aligned} \frac{r_k(\mathcal{P}_k(\hat{q}))}{\mathcal{P}_k(\hat{q})} &= \frac{\overline{v}_k(\hat{q})}{\overline{v}_k(q)} - 2 - \frac{1}{\overline{v}_k(q)\overline{v}_k(\hat{q})\left[\Gamma_k^*(\overline{v}_k(\hat{q})) - 1\right]} \left[\overline{v}_k(q)^2(1-q) - \int_0^q \overline{v}_k^2\right] \\ &= \frac{1-q}{1-\hat{q}} - 2 - \frac{1-\hat{q}}{\frac{\overline{v}_k}{\overline{v}_k'}(\hat{q}) - (1-\hat{q})} \left[1 - \frac{1}{\overline{v}_k(q)^2(1-q)} \int_0^q \overline{v}_k^2\right] \\ &= \frac{1-q}{1-\hat{q}} - 2 - \frac{1-\hat{q}}{-\frac{h}{h^+}(\hat{q}) - (1-\hat{q})} \left[1 - \frac{h_k(q)^2}{1-q} \int_0^q h_k^{-2}\right], \end{aligned}$$

which converges as $k \to \infty$ to $z := \frac{1-q}{1-\hat{q}} - 2 - \frac{1-\hat{q}}{-\frac{h}{h^+}(\hat{q})-(1-\hat{q})}\left[1 - \frac{h(q)^2}{1-q}\int_0^q h^{-2}\right]$ by the Lebesgue dominated convergence theorem. Hence, we can verify $r_k(\mathcal{P}_k(\hat{q})) > 0$ for sufficiently large k by verifying $z > 0$. Toward computing z, observe that

$h^+(\hat{q}) = \frac{-1}{1-\hat{q}}\frac{1}{\gamma}$ implies

$$\frac{1-\hat{q}}{-\frac{h}{h^+}(\hat{q}) - (1-\hat{q})} = \frac{1}{\gamma h(\hat{q}) - 1} = \frac{1}{\gamma}.$$

Furthermore, we can compute the integral,

$$\begin{aligned}
\int_0^q h^{-2} &= \int_0^q \left[h(\hat{q}) - h^-(\hat{q})(\hat{q}-\tilde{q})\right]^{-2} \, \mathrm{d}\tilde{q} \\
&= \left\{ \frac{-1}{h^-(\hat{q})} \left[h(\hat{q}) - h^-(\hat{q})(\hat{q}-\tilde{q})\right]^{-1} \right\}_{\tilde{q}=0}^{q} \\
&= \frac{1}{h^-(\hat{q})} \left[\frac{1}{h(\hat{q}) - h^-(\hat{q})\hat{q}} - \frac{1}{h(\hat{q}) - h^-(\hat{q})(\hat{q}-q)} \right] \\
&= \frac{q}{[h(\hat{q}) - h^-(\hat{q})\hat{q}]\,[h(\hat{q}) - h^-(\hat{q})(\hat{q}-q)]} \\
&= h(\hat{q})^{-2} \frac{q}{\left[1 + \frac{1}{1-\hat{q}}\hat{q}\right]\left[1 + \frac{1}{1-\hat{q}}(\hat{q}-q)\right]} \\
&= \left[\frac{1-\hat{q}}{h(\hat{q})}\right]^2 \frac{q}{[(1-\hat{q}) + \hat{q}]\,[(1-\hat{q}) + (\hat{q}-q)]} \\
&= \left[\frac{1-q}{h(q)}\right]^2 \frac{q}{1-q}.
\end{aligned}$$

Combining these calculations then yields

$$z = \frac{1-q}{1-\hat{q}} - 2 - \frac{1}{\gamma}(1-q) = \frac{3\gamma^2}{\gamma^2} - 2 - \frac{1}{\gamma}(3\gamma^2) = 1 - 3\gamma,$$

which is strictly positive as desired.

It remains to show that h can be approximated by some sequence in the sense above. To that end, first note that h it is continuous with a convex kink at $\hat{q}$, and has strictly negative derivative everywhere else. Continuity is obvious, and negativity of the derivative away from $\hat{q}$ follows from our formulae for one-sided derivatives at $\hat{q}$, because h is affine on either side of $\hat{q}$. To see the kink is convex, observe

$$h^+(\hat{q}) - h^-(\hat{q}) = \frac{1}{1-\hat{q}}\left[h(\hat{q}) - \frac{1}{\gamma}\right] = \frac{1}{1-\hat{q}} > 0.$$

Now, for any $\varepsilon \in (0, q)$, define the function $h_\varepsilon : (0, 1] \to \mathbb{R}$ by letting

$$h_\varepsilon(\tilde{q}) := \max\{h(\tilde{q}),\ \tfrac{\varepsilon}{\tilde{q}}\} + \varepsilon \min\{(\tilde{q} - q)(\tilde{q} - \hat{q}),\ (\tilde{q} - \hat{q})^2\}.$$

By construction, this function is continuous on $(0, 1]$ and strictly convex on $(0, \hat{q}]$ and $[\hat{q}, 1]$. Now, note $\tilde{q} \mapsto \frac{\varepsilon}{\tilde{q}}$ crosses h once from above, and the crossing point is below q. Indeed, because $h(\tilde{q}) \geq 1 > \frac{\varepsilon}{q} \geq \frac{\varepsilon}{\tilde{q}}$ for every $\tilde{q} \in [q, 1]$, we need only see that it crosses $h|_{(0,q]}$ once from above. This follows from the rankings around the endpoints $\{0, q\}$, the function $h|_{(0,q]}$ being affine, and $\tilde{q} \mapsto \frac{\varepsilon}{\tilde{q}}$ being convex. Letting q_ε denote the crossing point, the restrictions of h_ε to each of $(0, q_\varepsilon]$, $[q_\varepsilon, \hat{q}]$, and $[\hat{q}, 1]$ are all differentiable. Moreover, because h has a convex kink at $\hat{q}$ and a quadratic function has bounded derivative, sufficiently small ε will ensure that h_ε is has a convex kink at $\hat{q}$ (making it globally strictly convex) and has $h'_\varepsilon(1) < 0$. Also, h_ε has the same value as h at q and $\hat{q}$, and has the same right derivative at $\hat{q}$. Finally, h_ε converges pointwise to h as $\varepsilon \to 0$ and has $h_\varepsilon \geq \frac{1}{2}$ for small enough ε.

Now, take any $\varepsilon \in (0, q)$ small enough that h_ε is convex with $h'_\varepsilon(1) < 0$. For any $\delta \in \frac{1}{2}\min\{q_\varepsilon,\ \hat{q} - q\}$, let $h_{\varepsilon,\delta} : (0, 1] \to \mathbb{R}$ be a differentiable and strictly convex (hence continuously differentiable) function that agrees with h_ε outside of $(q_\varepsilon - \delta, q_\varepsilon) \cup (\hat{q} - \delta, \hat{q})$. For example, letting $\xi := \frac{h^+_\varepsilon(q_1) - h'_\varepsilon(q_1-\delta)}{h_\varepsilon(q_1) - h_\varepsilon(q_1-\delta) - \delta h'_\varepsilon(q_1-\delta)}$, set

$$\begin{aligned} h_{\varepsilon,\delta}(\tilde{q}) \quad := \quad & h_\varepsilon(q_1 - \delta) + [\tilde{q} - (q_1 - \delta)]\, h'_\varepsilon(q_1 - \delta) \\ & + \frac{1}{\xi}\left[h^+_\varepsilon(q_1) - h'_\varepsilon(q_1 - \delta)\right]\left[\frac{\tilde{q} - (q_1 - \delta)}{\delta}\right]^{\delta\xi} \end{aligned}$$

for every $q_1 \in \{q_\varepsilon, \hat{q}\}$ and $\tilde{q} \in (q_1 - \delta, q_1)$. Then, $(h_{\varepsilon,\delta})_\delta$ are uniformly bounded away from zero because h_ε is, and $h_{\varepsilon,\delta}$ converges pointwise to h_ε by construction.

Given the above arguments, there exist a sequence of functions $h_k := h_{\varepsilon_k,\delta_k}$ with $(\varepsilon_k, \delta_k) \to 0$ converging pointwise to h, and every function in the sequence is strictly convex and continuously differentiable with $\lim_{\tilde{q}\searrow 0} \tilde{q} h_k(\tilde{q}) \in (0, \infty)$, with the same values as h at q and $\hat{q}$ and the same right derivative as h at $\hat{q}$. Hence, $\overline{v}_k$, given by $\overline{v}_k(0) = 0$ and $\overline{v}_k|_{(0,1]} = 1/h_k$, is as required. *Q.E.D.*

Lemma 20. *Consider the linear demand model, and let (Π^*, q^*) be worst-case optimal. Let $p^N := \frac{1}{2}\overline{v}(q^*)$, and let p^* be the highest price in the support of Π^*. Then $p^* < p^N$ and $CS_{q^*}(p^N) < CS_{q^*}(\Pi^*)$.*

Proof. For the price ranking, note that Lemma 13 implies $0 = r(p^*) < \mathcal{V}(p^*) - 2p^* = \overline{v}(q^*) - 2p^* = 2(p^N - p^*)$, so $p^* < p^N$. Toward the consumer surplus ranking, observe that the map $[0, \overline{v}(q^*)] \to \mathbb{R}$ given by $p \mapsto \mathrm{CS}_{q^*}(p)$ is strictly decreasing, so that $\mathrm{CS}_{q^*}(p^N) < \mathrm{CS}_{q^*}(p^*) \leq \mathrm{CS}_{q^*}(\Pi^*)$.[53] *Q.E.D.*

D.4. Proof of Proposition 3

Observe the virtual value $\varphi := \varphi_{\overline{q},\overline{q}}$ is continuous with $\varphi(0) < 0 < \varphi(\overline{v}(\overline{q}))$, and Assumption 3 says it is strictly increasing on $[0, \overline{v}(\overline{q})]$. Hence, a unique $p^N \in (0, \overline{v}(q))$ exists with $\varphi(p^N) = 0$, and results from Myerson (1981) tell us the uniquely optimal price distribution for the no-externality case is the degenerate distribution Π^N on p^N.

Specialize from now on to the case of linear demand. In this case, any $v \in [0, \overline{v}(\overline{q})]$ has $\varphi(v) = 2v - \overline{v}(\overline{q})$, so that $p^N = \frac{1}{2}\overline{v}(\overline{q})$ and $q^N = \frac{1}{2}$. Hence,

$$q^* = D_{q^*}(\Pi^*) \geq D_{q^*}(p^*) > D_{q^*}\left(\tfrac{1}{2}\overline{v}(q^*)\right) = \tfrac{1}{2} = q^N,$$

where the strict inequality holds because Lemma 20 tells us $p^* < \frac{1}{2}\overline{v}(q^*) < \overline{v}(q^*)$.

We now turn to the price and consumer surplus rankings. Because multiplying $\overline{v}$ by a strictly positive multiple preserves the hypotheses of the linear demand model, Lemma 19 tells us there are instances of the linear demand

[53] Replacing Lemma 13 with Lemma 22, an identical argument shows $p^B < p^N$ and $\mathrm{CS}_{q^B}(p^N) < \mathrm{CS}_{q^B}(p^B)$ if $p^N = \frac{1}{2}\overline{v}(q^B)$.

model with $\overline{v}(1) = 1$ and $\overline{v}(q^*)$ arbitrarily close to zero. Because such a model has $p^N = \frac{1}{2}$ and $\text{CS}_{\overline{q}}(p^N) = \frac{1}{8}\overline{q}$ whereas p^* and $\text{CS}_{q^*}(\Pi^*)$ are both $\leq \overline{v}(q^*)$, it follows that $p^N > p^*$ and $\text{CS}_{\overline{q}}(p^N) > \text{CS}_{q^*}(\Pi^*)$ are possible. Conversely, fixing any $\overline{v}$, note that p^N and $\text{CS}_{\overline{q}}(p^N)$ are both $\leq \overline{v}(\overline{q})$; hence, taking $\overline{q}$ close enough to zero witnesses that $p^N < p^*$ and $\text{CS}_{\overline{q}}(p^N) < \text{CS}_{q^*}(\Pi^*)$ are possible.

Finally, let us establish the last assertion—that some $\overline{v}$, with $\overline{q} = 1$, has $p^N > p^*$ and $\text{CS}_{\overline{q}}(p^N) < \text{CS}_{q^*}(\Pi^*)$. Indeed, consider our leading example with $\overline{v}(q) = q$ for every $q \in [0, 1]$. For this example, Lemma 15 tells us $p^* < \frac{1}{2} = p^N$ yields consumer surplus $\text{CS}^* := \text{CS}_{q^*}(\Pi^*)$ with externalities such that $\text{CS}^* > \frac{1}{6} > \frac{1}{8}$. The associated no-externalities consumer surplus is

$$\text{CS}_{\overline{q}}(p^N) = \text{CS}_1(\tfrac{1}{2}) = \int_{\frac{1}{2}}^{1} (1 - v)\ \mathrm{d}v = \tfrac{1}{8} < \text{CS}^*.$$

The proposition follows. *Q.E.D.*

D.5. Inputs for the proof of Proposition 4

Lemma 21. *Under best-case equilibrium selection, an optimal price distribution exists and yields strictly positive revenue, and any such distribution is degenerate.*

Proof. Recall, every price distribution admits a best-case equilibrium quantity (Lemma 2). It follows that a price distribution is best-case optimal if and only if, paired with some quantity, it solves the program

$$\sup_{\Pi \in \Delta(\mathbb{R}_+),\ q \in [0,1]} R_q(\Pi) \text{ subject to } D_q(\Pi) = q.$$

If $\Pi \in \Delta(\mathbb{R}_+)$ and $q \in [0, 1]$ satisfy the weaker constraint $D_q(\Pi) \geq q$, then (given Lemma 2) the Knaster-Tarski theorem delivers an equilibrium quantity for Π which is weakly higher and hence generates a weakly higher revenue. So a price distribution is best-case optimal if and only if, paired with some quantity, it solves the program

$$\sup_{\Pi\in\Delta(\mathbb{R}_+),\ q\in[0,1]} R_q(\Pi) \ \text{ subject to } \ D_q(\Pi) \geq q. \tag{P_B^*}$$

In what follows, let us work with the latter program.

Let us first establish existence of an optimum. Replacing any price distribution Π with the modified distribution $\Pi|_{[0,\overline{v}(1))} \cup \mathbf{1}|_{[\overline{v}(1),\infty)} \in \Delta[0,\overline{v}(1)]$ leaves the demand and revenue unchanged, and so it suffices to see the program with price distributions restricted to $\Delta[0,\overline{v}(1)]$ has an optimum. And indeed, Lemma 3 tells us this program has compact domain and upper semicontinuous objective, and so admits an optimum.

Consider now any feasible pair (Π, q) from program (P_B^*) with Π nondegenerate; let us show some price $\hat{p} \in \mathbb{R}_+$ exists such that the degenerate price distribution on $\hat{p}$, paired with some quantity $\hat{q}$, is also feasible in program (P_B^*) and with $R_{\hat{q}}(\hat{p}) > R_q(\Pi)$. If $q = 0$, then $R_q(\Pi) = 0$, and so any quantity $\hat{q} \in (0,1)$ paired with (the degenerate price distribution on) sufficiently small $\hat{p} > 0$ will be feasible in the program and yield strictly higher revenue. Focus now on the case of $q > 0$, and take $\hat{q} := q$. Observe $p \mapsto D_q(p)$ is strictly decreasing and continuous on $[0, \overline{v}(q)]$ with $D_q(0) = 1 > q > 0 > D_q(\overline{v}(q))$. So the intermediate value theorem tells us $\hat{p} := D_q^{-1}(q) \in (0, \overline{v}(q))$. By construction, $\hat{q} = q$ is an equilibrium quantity for the degenerate distribution on price $\hat{p}$. Moreover, because $\varphi_{q,q}$ is strictly increasing (given Assumption 3), the degenerate price yields a strictly higher revenue. *Q.E.D.*

Lemma 22. *Take the linear demand environment.*

(i) If the seller posts a price strictly greater than $\overline{p}^B := \int_0^{\overline{p}^*}(1-\Gamma^*) \in (0, \overline{p}^*)$, *then the highest equilibrium quantity is zero.*

(ii) If the seller posts a price $\hat{p} \in [0, \overline{p}^B]$, *then a given* $\hat{q}$ *is an equilibrium quantity if and only if* $\hat{p} = \mathcal{P}^B(\hat{q})$, *where* $\mathcal{P}^B(\hat{q}) := \int_0^{\overline{v}(\hat{q})}(1-\Gamma^*)$. *In particular, the highest such quantity is the unique* $\mathcal{Q}^B(\hat{p}) \in [\underline{q}(\overline{p}^*), 1]$ *such that* $\mathcal{P}^B(\mathcal{Q}^B(\hat{p})) = \hat{p}$.

(iii) The functions $\mathcal{P}^B$ *and* $\mathcal{Q}^B$ *are continuously differentiable on* $(\underline{q}(\overline{p}^*), 1]$

and $[0,\overline{p}^B)$, *respectively. Any* $\hat{p} \in [0,\overline{p}^B)$ *and* $\hat{q} = \mathcal{Q}^B(\hat{p}) \in (\underline{q}(\overline{p}^*), 1]$ *have*

$$\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\mathcal{P}^B(\hat{q}) = -\overline{v}'(\hat{q})\left[\Gamma^*(\overline{v}(\hat{q})) - 1\right] \text{ and } \frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{Q}^B(\hat{p}) = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\Big|_{\hat{q}=\mathcal{Q}^B(\hat{p})}\mathcal{P}^B(\hat{q})},$$

which are both strictly negative.

(iv) Letting $\mathcal{R}^B(\hat{p},\hat{q}) := R_{\hat{q}}(\hat{p})$, *any* $\hat{p} \in [0,\overline{p}^B)$ *has* $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}^B\left(\hat{p}, \mathcal{Q}^B(\hat{p})\right) = \frac{1}{\overline{v}(\mathcal{Q}^B(\hat{p}))}r^B(\hat{p})$, *where*

$$r^B(\hat{p}) := \left[\overline{v}(\mathcal{Q}^B(\hat{p})) - 2\hat{p}\right] - \frac{1}{\overline{v}(\mathcal{Q}^B(\hat{p}))\left[\Gamma^*(\overline{v}(\mathcal{Q}^B(\hat{p}))) - 1\right]}\,\hat{p}^2.$$

(v) The function $r^B : [0,\overline{p}^B) \to \mathbb{R}$ *is continuously differentiable with strictly negative derivative.*

(vi) The function $[0,\overline{p}^B] \to \mathbb{R}$ *given by* $\hat{p} \mapsto \mathcal{R}^B\left(\hat{p}, \mathcal{Q}^B(\hat{p})\right)$ *is strictly quasiconcave with interior maximum.*

(vii) Under best-case equilibrium selection, the unique optimal price distribution is degenerate at the unique price $p^B \in (0,\overline{p}^B)$ *with* $r^B(p^B) = 0$, *and the unique best equilibrium quantity at that price is* $q^B := \mathcal{Q}^B(p^B)$.

Proof. Lemma 21 says the seller optimally chooses a deterministic price and so solves

$$\max_{\hat{p}\in\mathbb{R}_+,\ \hat{q}\in[0,1]} \mathcal{R}^B(\hat{p},\hat{q}) \text{ s.t. } D_{\hat{q}}(\hat{p}) = \hat{q}.$$

By the same lemma, an optimum exists and has both price and quantity being strictly positive. Now, let us rewrite the equilibrium constraint. Any quantity $\hat{q} \in (0,1]$ and price $\hat{p} \in \mathbb{R}_+$ have

$$D_{\hat{q}}(\hat{p}) = \hat{q} \iff 1 - \tfrac{\hat{p}}{\overline{v}(\hat{q})} = \hat{q} \iff \hat{p} = (1-\hat{q})\overline{v}(\hat{q}) \iff \hat{p} = \int_0^{\overline{v}(\hat{q})}(1-\Gamma^*),$$

where the last equivalence follows from Lemma 12. Now, defining $\mathcal{P}^B : [0,1] \to \mathbb{R}$ given by $\mathcal{P}^B(\hat{q}) := \int_0^{\overline{v}(\hat{q})}(1-\Gamma^*)$, Lemma 12 tells us $\mathcal{P}^B$ is continuous and strictly quasiconcave with $\mathcal{P}^B(0) = \mathcal{P}^B(1) = 0$ and maximizer $\underline{q}(\overline{p}^*)$.

Therefore, the range of $\mathcal{P}^B$ is $[0, \overline{p}^B]$ for $\overline{p}^B := \int_0^{\overline{p}^*}(1-\Gamma^*) \in (0, \overline{p}^*)$, and every $\hat{p} \in [0, \overline{p}^B]$ has one solution in $[0, \underline{q}(\overline{p}^*)]$ and one solution in $[\underline{q}(\overline{p}^*), 1]$ to $\mathcal{P}^B(\cdot) = \hat{p}$. So let $\mathcal{Q}^B : [0, \overline{p}^B] \to [\underline{q}(\overline{p}^*), 1]$ be such that $\mathcal{P}^B(\mathcal{Q}^B(\hat{p})) = \hat{p}$ for every $\hat{p} \in [0, \overline{p}^B]$; Lemma 4 tells us seller revenue is strictly increasing (wherever strictly positive, as the optimal revenue is) in the quantity, and so the best equilibrium quantity if the seller posts price $\hat{p}$ is $\hat{q} = \mathcal{Q}^B(\hat{p})$. We can thus write the seller's problem under best case selection as

$$\max_{\hat{p} \in [0, \overline{p}^B]} \mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p})).$$

Now, the function $\mathcal{Q}^B$ is continuous and strictly decreasing by construction. Moreover, because $\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\mathcal{P}^B(\hat{q}) = -\overline{v}'(\hat{q})\,[\Gamma^*(\overline{v}(\hat{q})) - 1]$, which is continuous and strictly negative for $\hat{q} \in (\underline{q}(\overline{p}^*), 1]$, the inverse function theorem tells us $\mathcal{Q}^B$ is continuously differentiable on $[0, \overline{p}^B)$ with derivative $\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{Q}^B(\hat{p}) = \frac{-1}{\overline{v}'(\mathcal{Q}^B(\hat{p}))[\Gamma^* \circ \overline{v} \circ \mathcal{Q}^B(\hat{p}) - 1]}$ there. We are now equipped to compute the seller's first-order condition under best-case selection. For any $\hat{p} \in (0, \overline{p}^B)$, at $\hat{q} = \mathcal{Q}^B(\hat{p})$ and $\hat{v} = \overline{v}(\hat{q})$ we have

$$\begin{aligned}
\overline{v}(\mathcal{Q}^B(\hat{p}))\ \frac{\mathrm{d}}{\mathrm{d}\hat{p}}\mathcal{R}^B(\hat{p}, \mathcal{Q}^B(\hat{p})) &= \hat{v}\left[\frac{\partial}{\partial\hat{p}}\mathcal{R}^B(\hat{p}, \hat{q}) + (\mathcal{Q}^B)'(\hat{p})\ \frac{\partial}{\partial\hat{q}}\mathcal{R}^B(\hat{p}, \hat{q})\right] \\
&= \hat{v}\left\{\frac{\partial}{\partial\hat{p}}\left[\hat{p}\left(1 - \frac{\hat{p}}{\overline{v}(\hat{q})}\right)\right] + (\mathcal{Q}^B)'(\hat{p})\ \frac{\partial}{\partial\hat{q}}\left[\hat{p}\left(1 - \frac{\hat{p}}{\overline{v}(\hat{q})}\right)\right]\right\} \\
&= \hat{v}\left\{\left[1 - \frac{2\hat{p}}{\hat{v}}\right] + \frac{-1}{\overline{v}'(\hat{q})\,[\Gamma^*(\hat{v}) - 1]}\ \frac{\hat{p}^2}{\hat{v}^2}\overline{v}'(\hat{q})\right\} \\
&= [\hat{v} - 2\hat{p}] - \frac{1}{\hat{v}\,[\Gamma^*(\hat{v}) - 1]}\ \hat{p}^2 \\
&= \left[\overline{v}(\mathcal{Q}^B(\hat{p})) - 2\hat{p}\right] - \frac{1}{\overline{v}(\mathcal{Q}^B(\hat{p}))\,[\Gamma^*(\overline{v}(\mathcal{Q}^B(\hat{p}))) - 1]}\ \hat{p}^2,
\end{aligned}$$

which is $r^B(\hat{p})$. Because the denominator $\overline{v}(\mathcal{Q}^B(\hat{p}))\left[\Gamma^*(\overline{v}(\mathcal{Q}^B(\hat{p}))) - 1\right]$ is positive and decreasing in $\hat{p} \in (0, \overline{p}^B)$, it follows that the function $\hat{p} \mapsto r^B(\hat{p}) + 2\hat{p}$ is decreasing on $(0, \overline{p}^B)$, and so r^B has strictly negative derivative there—hence the seller's problem under best-case selection is strictly quasiconcave in the choice of posted price. To see an interior root of r^B exists, note first that

$r(0) = \overline{v}(1) > 0$. Meanwhile that $\Gamma^*(\overline{v}(\mathcal{Q}^B(\hat{p}))) - 1 \searrow 0$ as $\hat{p} \nearrow \overline{p}^B$ tells us $r^B(\hat{p}) \to -\infty$. Hence, any $\hat{p} \in (0, \overline{p}^B)$ close enough to $\overline{p}^B$ has $r^B(\hat{p}) < 0$; thus the intermediate value theorem then applies. *Q.E.D.*

D.6. Proof of Proposition 4

Given Lemma 22, we know the strictly positive price $p^B < \overline{p}^B < \overline{p}^*$ is the unique price such that $r^B(p^B) = 0$, and $q^B = \mathcal{Q}^B(p^B)$. We will use these facts to compare with worst-case selection. Before doing so, note that strictly positive revenue (assured by Lemma 21) implies $0 < q^B < 1$ and $0 < p^B < \overline{v}(q^B)$.

Let p^* and q^* be the highest supported price and equilibrium quantity as described in Lemma 14. We want to show that $p^* > p^B$ and $q^* > q^B$. Because $\mathcal{Q}$ is strictly decreasing, we can equivalently show that $\mathcal{P}(q^B) > p^* > p^B$. By Lemma 13, we can rewrite this condition as the requirement that $r(\mathcal{P}(q^B)) < 0 < r(p^B)$. Toward both of these rankings, observe that any $\hat{p} \in (0, \overline{p}^*)$ has

$$\begin{aligned} r(\hat{p}) &= r(\hat{p}) - r^B(p^B) \\ &= [\mathcal{V}(\hat{p}) - 2\hat{p}] - \left[\overline{v}(q^B) - 2p^B\right] \\ &\quad - \tfrac{1}{\mathcal{V}(\hat{p})[\Gamma^*(\mathcal{V}(\hat{p}))-1]} \int_0^\infty p^2 \, \mathrm{d}\Pi(p|\hat{p}) + \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \, (p^B)^2 \end{aligned}$$

Toward the price ranking, note that specializing the above calculation yields

$$r(p^B) = \mathcal{V}(p^B) - \overline{v}(q^B) - \tfrac{1}{\mathcal{V}(p^B)[\Gamma^*(\mathcal{V}(p^B))-1]} \int_0^\infty p^2 \, \mathrm{d}\Pi(p|p^B) + \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \, (p^B)^2.$$

That $\Pi(p^B|p^B) = 1$ implies $\int_0^\infty p^2 \, \mathrm{d}\Pi(p|p^B) \leq (p^B)^2$; and that Γ^* is increasing (Lemma 12) and $\mathcal{V}$ decreasing implies $\frac{1}{\mathcal{V}\cdot[\Gamma^* \circ \mathcal{V} - 1]}$ is increasing on $[0, \overline{p}^*)$. Hence,

$r(p^B) > 0$ will follow if we show $\mathcal{V}(p^B) > \overline{v}(q^B)$. And indeed,

$$
\begin{aligned}
\int_{\overline{v}(q^B)}^{\mathcal{V}(p^B)} (\Gamma^* - 1) &= \left(\int_0^{\overline{v}(q^B)} - \int_{p^B}^{\mathcal{V}(p^B)} - \int_0^{p^B} \right) (1 - \Gamma^*) \\
&= p^B - 0 - \int_0^{p^B} (1 - \Gamma^*) \\
&= \int_0^{p^B} \Gamma^* > 0,
\end{aligned}
$$

delivering $\mathcal{V}(p^B) > \overline{v}(q^B)$ because $\Gamma^* > 1$ between them. The price ranking follows.

Toward the quantity ranking, observe that

$$
\begin{aligned}
r(\mathcal{P}(q^B)) &= \left[\mathcal{V}(\mathcal{P}(q^B)) - 2\mathcal{P}(q^B)\right] - \left[\overline{v}(q^B) - 2p^B\right] \\
&\quad - \tfrac{1}{\mathcal{V}(\mathcal{P}(q^B))[\Gamma^*(\mathcal{V}(\mathcal{P}(q^B)))-1]} \int_0^\infty p^2 \, \mathrm{d}\Pi(p|\mathcal{P}(q^B)) + \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \, (p^B)^2 \\
&= -2\left[\mathcal{P}(q^B) - p^B\right] - \tfrac{1}{\overline{v}(q^B)[\Gamma^*(\overline{v}(q^B))-1]} \left[\int_0^\infty p^2 \, \mathrm{d}\Pi(p|\mathcal{P}(q^B)) - (p^B)^2 \right].
\end{aligned}
$$

So $r(\mathcal{P}(q^B)) < 0$ would follow if we knew $\mathcal{P}(q^B) > p^B$ and $\int_0^\infty p^2 \, \mathrm{d}\Pi(p|\mathcal{P}(q^B)) \geq (p^B)^2$. To establish both inequalities, observe that

$$
p^B = p^B - 0 = \left(\int_0^{\overline{v}(q^B)} - \int_{\mathcal{P}(q^B)}^{\mathcal{V}(\mathcal{P}(q^B))} \right) (1 - \Gamma^*) = \int_0^{\mathcal{P}(q^B)} (1 - \Gamma^*).
$$

This identity first implies $\mathcal{P}(q^B) > p^B$ because $\int_0^{\mathcal{P}(q^B)}(1 - \Gamma^*) < \mathcal{P}(q^B)$. Then, to show $\int_0^\infty p^2 \, \mathrm{d}\Pi(p|\mathcal{P}(q^B)) \geq (p^B)^2$, it suffices to show $\int_0^\infty p^2 \, \mathrm{d}\Pi(p|\hat{p}) - \left[\int_0^{\hat{p}}(1 - \Gamma^*)\right]^2$ is nonnegative for any $\hat{p} \in [0, \overline{p}^*]$. And indeed, the expression is obviously zero for $\hat{p}=0$, and it satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\left\{\int_0^\infty p^2 \,\mathrm{d}\Pi(p|\hat{p}) - \left[\int_0^{\hat{p}}(1-\Gamma^*)\right]^2\right\} &= \frac{\mathrm{d}}{\mathrm{d}\hat{p}}\left\{2\int_0^{\hat{p}} p\left[1-\Gamma^*(p)\right]\,\mathrm{d}p - \left[\int_0^{\hat{p}}(1-\Gamma^*)\right]^2\right\}\\ &= 2\hat{p}\left[1-\Gamma^*(\hat{p})\right] - 2\left[\int_0^{\hat{p}}(1-\Gamma^*)\right]\left[1-\Gamma^*(\hat{p})\right]\\ &= 2\left[1-\Gamma^*(\hat{p})\right]\int_0^{\hat{p}}\Gamma^* \geq 0.\end{aligned}$$

The quantity ranking follows.

Finally, we turn to the consumer surplus ranking. Let $\Pi^* := \Pi(\cdot|p^*)$ be the optimal (under worst-case selection) price distribution characterized in Lemma 14, and let Π be the modified price distribution given by capping the price at $\overline{v}(q^B)$—that is $\Pi(p)$ is equal to $\Pi^*(p)$ for $p < \overline{v}(q^B)$, and is equal to 1 for $p \geq \overline{v}(q^B)$. Observe, any $p \in \mathbb{R}_+$ has

$$\begin{aligned}\tfrac{1}{\overline{v}(q^B)}\left[p^B - \min\{p,\ \overline{v}(q^B)\}\right] &= D_{q^B}(\min\{p,\ \overline{v}(q^B)\}) - D_{q^B}(p^B)\\ &= D_{q^B}(p) - q^B,\end{aligned}$$

and so $\frac{1}{\overline{v}(q^B)}\left[p^B - \int_0^\infty p\,\mathrm{d}\Pi(p)\right] = D_{q^B}(\Pi^*) - q^B$, which is nonnegative because $q^B < q^*$ and (Π^*, q^*) satisfies the demand constraints. Having established $p^B \geq \int_0^\infty p\,\mathrm{d}\Pi(p)$, we now pursue the surplus ranking. To that end, define

$$\mathrm{CS}_q(p) := \int (v-p)_+ \,\mathrm{d}F_q(v) = \int_p^\infty D_q$$

the consumer surplus associated with anticipated quantity q (hence demand curve D_q) and a price offer of p; and let $\mathrm{CS}_q(\hat{\Pi}) := \int \mathrm{CS}_q(p)\,\mathrm{d}\hat{\Pi}(p)$ for any price distribution $\hat{\Pi}$. Observe that $\mathrm{CS}_q(p)$ is decreasing in p, strictly convex in p (because D_q is strictly decreasing) wherever $0 \leq p \leq \overline{v}(q)$, and (because $u(\theta,\cdot)$ is increasing) increasing in q. Moreover, the price distribution Π is nondegenerate because $p^B > 0$ and (because $p^* > 0$ and Lemma 13 says Γ^* is strictly increasing) Π^* is strictly increasing in a neighborhood of zero.

Therefore,

$$\mathrm{CS}_{q^B}(p^B) \le \mathrm{CS}_{q^B}\left(\int p \, \mathrm{d}\Pi(p)\right) < \mathrm{CS}_{q^B}(\Pi) = \mathrm{CS}_{q^B}(\Pi^*) \le \mathrm{CS}_{q^*}(\Pi^*),$$

where the last inequality holds because $q^* \ge q^B$. *Q.E.D.*

E. Proofs for Section 5

E.1. Proof of Proposition 5

For any $\omega \in [0,1]$, define $\overline{v}_\omega : [0,1] \to \mathbb{R}$ by letting $\overline{v}_\omega(q) := \frac{1}{(1-\omega)\frac{1}{\overline{v}_0(q)} + \omega \frac{1}{\overline{v}_1(q)}}$ for $q \in (0,1]$, and $\overline{v}_\omega(0) := 0$; this $\overline{v}_\omega$ is also an instance of the linear demand environment. In particular, $\frac{1}{\overline{v}_\omega}$ inherits strict convexity from $\frac{1}{\overline{v}_0}$ and $\frac{1}{\overline{v}_1}$. Observe that any $q \in (0,1]$ has

$$\frac{\partial}{\partial \omega} \log \overline{v}_\omega(q) = -\frac{\partial}{\partial \omega} \log \frac{1}{\overline{v}_\omega(q)} = -\frac{\frac{1}{\overline{v}_1(q)} - \frac{1}{\overline{v}_0(q)}}{\frac{1}{\overline{v}_\omega(q)}} = \frac{\frac{\overline{v}_1(q)}{\overline{v}_0(q)} - 1}{(1-\omega)\frac{\overline{v}_1(q)}{\overline{v}_0(q)} + \omega},$$

which is strictly increasing in q because $\frac{\overline{v}_1(q)}{\overline{v}_0(q)}$ is. Equivalently, whenever $0 < q < \tilde{q} \le 1$, we have $\frac{\partial}{\partial \omega}\left[\frac{\overline{v}_\omega(\tilde{q})}{\overline{v}_\omega(q)}\right] > 0$, a log-supermodularity property that will be useful in establishing the quantity ranking.

First, we pursue the price distribution ranking. That $\frac{\overline{v}_1}{\overline{v}_0}$ has nonnegative derivative on $(0,1]$ means $\frac{\overline{v}_1}{\overline{v}_1'} \le \frac{\overline{v}_0}{\overline{v}_0'}$, and so $\Gamma_1^* \circ \overline{v}_1 \le \Gamma_0^* \circ \overline{v}_0$. Using this fact, let us see that $\Gamma_1^*(p) < \Gamma_0^*(p)$ for any price p with $0 < p \le \min\{\overline{v}_0(1), \overline{v}_1(1)\}$. To see it, let $q_\omega := \overline{v}_\omega^{-1}(p) \in (0,1]$ for each $\omega \in \{0,1\}$. That $\overline{v}_1$ exhibits stronger externalities than $\overline{v}_0$ implies $q_1 < q_0$—since the strictly increasing function $\frac{\overline{v}_1}{\overline{v}_0}$ is above 1 on $(0,1]$, it is in fact strictly above 1 there.[54] It follows that

$$\Gamma_0^*(p) = \Gamma_0^*(\overline{v}_0(q_0)) \ge \Gamma_1^*(\overline{v}_1(q_0)) > \Gamma_1^*(\overline{v}_1(q_1)) = \Gamma_1^*(p).$$

Therefore, given Lemma 14, any p with $0 < p < \min\{p_1^*, p_0^*\}$ has $\Pi_0^*(p) =$

[54] This observation is the only part of the proposition that uses the condition that $\overline{v}_1 \ge \overline{v}_0$. In particular, the quantity ranking follows only from $\frac{\overline{v}_1}{\overline{v}_0}$ being strictly increasing on $(0,1]$.

$\Gamma_0^*(p) > \Gamma_1^*(p) = \Pi_1^*(p)$.

Next, we turn to the quantity ranking. Define $\tilde{r}(\hat{q}, \omega) := \frac{r_\omega(\mathcal{P}_\omega(\hat{q}))}{\mathcal{P}_\omega(\hat{q})}$ for any $(\hat{q}, \omega) \in [0, 1) \times [0, 1]$ with $\overline{v}_\omega(\hat{q}) > \overline{p}_\omega^*$. Below, we will show that the function $\tilde{r}$ has strictly negative partial derivative with respect to its second argument at (q_ω^*, ω) for any $\omega \in [0, 1]$; let us now see that doing so would establish the result. First, an application of the implicit function theorem tells us $(\hat{q}, \omega) \mapsto \mathcal{P}_\omega(\hat{q})$ is continuously differentiable on the range of $(\hat{q}, \omega) \in [0, 1) \times [0, 1]$ with $\overline{v}_\omega(\hat{q}) > \overline{p}_\omega^*$, and that $\mathcal{P}_\omega$ is strictly decreasing (Lemma 13) tells us it is strictly positive there. Because $(\hat{p}, \omega) \mapsto r_\omega(\hat{p})$ is continuously differentiable on the range of $(\hat{p}, \omega) \in \mathbb{R}_+ \times [0, 1]$ with $\hat{p} < \overline{p}_\omega^*$, it follows that $\tilde{r}$ is continuously differentiable on its domain. Next, observe, any $\omega \in [0, 1]$ has

$$\frac{\partial}{\partial \hat{q}}\bigg|_{\hat{q}=q_\omega^*} \tilde{r}(\hat{q}, \omega) = \frac{p_\omega^* r_\omega'(p_\omega^*) - 1 r_\omega(p_\omega^*)}{(p_\omega^*)^2} \mathcal{P}_\omega'(q_\omega^*) = \frac{\mathcal{P}_\omega'(q_\omega^*)}{p_\omega^*} r_\omega'(p_\omega^*) > 0,$$

where the second equality holds by Lemma 13 and Lemma 14 and the strict inequality follows from Lemma 13. Because q_ω^* is the unique solution $\hat{q}$ to $\tilde{r}(\hat{q}, \omega) = 0$ for each $\omega \in [0, 1]$, it follows that $\omega \mapsto q_\omega^*$ is continuously differentiable. Therefore, at any $\omega \in [0, 1]$ and $\hat{q} = q_\omega^*$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\omega} 0 = \frac{\mathrm{d}}{\mathrm{d}\omega} \tilde{r}(q_\omega^*, \omega) = \left[\frac{\partial}{\partial \omega} \tilde{r}(\hat{q}, \omega)\right] + \left[\frac{\partial}{\partial \hat{q}} \tilde{r}(\hat{q}, \omega)\right] \left[\frac{\partial}{\partial \omega} q_\omega^*\right].$$

Because we have shown $\frac{\partial}{\partial \hat{q}}\big|_{\hat{q}=q_\omega^*} \tilde{r}(\hat{q}, \omega) > 0$, the hypothesis that $\frac{\partial}{\partial \omega}\big|_{\hat{q}=q_\omega^*} \tilde{r}(\hat{q}, \omega) < 0$ therefore implies $\frac{\partial}{\partial \omega} q_\omega^* > 0$; hence, $\omega \mapsto q_\omega^*$ is strictly increasing, and so $q_1^* > q_0^*$.

Thus, all that remains is to show is that $\frac{\partial}{\partial \omega} \tilde{r}(\hat{q}, \omega) < 0$ wherever $\tilde{r}(\hat{q}, \omega)$ is zero. To that end, fix any $q^* \in (0, 1)$ for the remainder of our analysis. Define now the continuously differentiable (by Lemma 12 and Lemma 13) functions

$$\begin{aligned} r^* : (0,1)\times[0,1] &\to \mathbb{R} \\ (q,\omega) &\mapsto \frac{1-q}{1-q^*} - 2 - \frac{2(1-q^*)}{\overline{v}_\omega(q)^2(1-q)\left[\Gamma^*_\omega(\overline{v}_\omega(q^*))-1\right]}\int_0^{\overline{v}_\omega(q)} p\left[1-\Gamma^*_\omega(p)\right]\mathrm{d}p \\ &= \frac{1-q}{1-q^*} - 2 - \frac{1-q^*}{\Gamma^*_\omega(\overline{v}_\omega(q^*))-1}\left[1 - \frac{1}{\overline{v}_\omega(q)^2(1-q)}\int_0^q \overline{v}_\omega^2\right], \end{aligned}$$

where the equality follows from Lemma 13(ii). Next, define

$$\begin{aligned} q_* : \{\omega\in[0,1]:\ \overline{v}_\omega(q^*) > \overline{p}^*_\omega\} &\to (0,1) \\ \omega &\mapsto \overline{v}_\omega^{-1}\left(\mathcal{P}_\omega(q^*)\right). \end{aligned}$$

Now, for any ω in the domain of q_*, the definition of $\mathcal{P}_\omega$ implies

$$0 = \int_{\overline{v}_\omega(q_*(\omega))}^{\overline{v}_\omega(q^*)}(1-\Gamma^*) = \int_0^{\overline{v}_\omega(q^*)}(1-\Gamma^*) - \int_0^{\overline{v}_\omega(q_*(\omega))}(1-\Gamma^*) = (1-q^*)\overline{v}_\omega(q^*) - \left[1-q_*(\omega)\right]\overline{v}_\omega(q_*(\omega)),$$

where the last identity follows from Lemma 12. It follows that every such ω has $\tilde{r}(q^*,\omega) = r^*(q_*(\omega),\omega)$. Therefore,

$$\frac{\partial}{\partial\omega}\tilde{r}(q^*,\omega) = \left[\frac{\partial}{\partial q}\bigg|_{q=q_*(\omega)} r^*(q,\omega)\right] q_*'(\omega) + \left[\frac{\partial}{\partial\omega}\bigg|_{q=q_*(\omega)} r^*(q,\omega)\right].$$

To show $\frac{\partial}{\partial\omega}\tilde{r}(q^*,\omega) < 0$, it thus suffices to show $\frac{\partial}{\partial q}\big|_{q=q_*(\omega)} r^*(q,\omega)$ and $\frac{\partial}{\partial\omega}\big|_{q=q_*(\omega)} r^*(q,\omega)$ are both strictly negative and $q_*'(\omega)$ is strictly positive. We pursue each of these three inequalities.

Toward signing $q_*'(\omega)$, observe the definition of $q_*(\cdot)$ and Lemma 12 imply

$$0 = \int_{\overline{v}_\omega(q_*(\omega))}^{\overline{v}_\omega(q^*)}(1-\Gamma^*) = (1-q^*)\overline{v}_\omega(q^*) - \left[1-q_*(\omega)\right]\overline{v}_\omega(q_*(\omega)),$$

which rearranges to

$$1-q^* = \left[1-q_*(\omega)\right]\frac{\overline{v}_\omega(q_*(\omega))}{\overline{v}_\omega(q^*)}.$$

Differentiating the above equation tells us

$$\begin{aligned} 0 &= [1-q_*(\omega)]\,\frac{\partial}{\partial\omega}\bigg|_{q=q_*(\omega)}\left[\frac{\overline{v}_\omega(q)}{\overline{v}_\omega(q^*)}\right] + q_*'(\omega)\frac{\partial}{\partial q}\bigg|_{q=q_*(\omega)}\left[(1-q)\,\frac{\overline{v}_\omega(q)}{\overline{v}_\omega(q^*)}\right] \\ &< q_*'(\omega)\frac{\partial}{\partial q}\bigg|_{q=q_*(\omega)}\left[(1-q)\,\frac{\overline{v}_\omega(q)}{\overline{v}_\omega(q^*)}\right] \\ &= q_*'(\omega)\frac{\overline{v}_\omega'(q_*(\omega))}{\overline{v}_\omega(q^*)}\left[1-\Gamma^*(\overline{v}_\omega(q_*(\omega)))\right]. \end{aligned}$$

As $\Gamma^*(\overline{v}_\omega(q_*(\omega))) < 1$ by Lemma 12, it follows that $q_*'(\omega) > 0$.

Now, to sign $\frac{\partial}{\partial\omega}r^*(q,\omega)$, it suffices to show

$$\frac{1-\frac{1}{\overline{v}_\omega(q)^2(1-q)}\int_0^q \overline{v}_\omega^2}{\Gamma_\omega^*(\overline{v}_\omega(q^*))-1}$$

has strictly positive partial derivative with respect to ω at $q = q_*(\omega)$. Because (given Lemma 13(ii) and Lemma 12, respectively) both the numerator and denominator are strictly positive there, it suffices to show the numerator has positive partial derivative and denominator has negative partial derivative with respect to ω, at least one of them strictly so. First, the numerator's partial derivative is a strictly negative multiple of

$$\frac{\partial}{\partial\omega}\left[\frac{\int_0^q \overline{v}_\omega^2}{\overline{v}_\omega(q)^2}\right] = \frac{\partial}{\partial\omega}\int_0^q\left[\frac{\overline{v}_\omega(\hat{q})}{\overline{v}_\omega(q)}\right]^2\,\mathrm{d}\hat{q} = 2\int_0^q \frac{\overline{v}_\omega(\hat{q})}{\overline{v}_\omega(q)}\,\frac{\partial}{\partial\omega}\left[\frac{\overline{v}_\omega(\hat{q})}{\overline{v}_\omega(q)}\right]\,\mathrm{d}\hat{q},$$

which is strictly negative. Second, the denominator's partial derivative is

$$\frac{\partial}{\partial\omega}\left[\Gamma_\omega^*(\overline{v}_\omega(q^*))-1\right] = \frac{\partial}{\partial\omega}\left[\frac{\overline{v}_\omega(q^*)}{\overline{v}_\omega'(q^*)}\right] = \frac{\partial}{\partial\omega}\left\{\frac{1}{\frac{\partial}{\partial q}\big|_{q=q^*}\log\overline{v}_\omega(q)}\right\},$$

which is nonpositive by log-supermodularity. So $\frac{\partial}{\partial\omega}r^*(q,\omega) < 0$.

Finally, we turn to signing $\frac{\partial}{\partial q}r^*(q,\omega)$. To that end, let $q := q_*(\omega)$, let $v := \overline{v}_\omega(q)$, and let $v' := \overline{v}_\omega'(q)$. Then, that $\Gamma_\omega^*(v) < 1$ rearranges to $(1-q)v' > v$.

Hence,

$$\begin{aligned}
\frac{\partial}{\partial q}\left[\frac{\int_0^q \overline{v}_\omega^2}{\overline{v}_\omega(q)^2(1-q)}\right] &= \frac{v^2(1-q)v^2 - [2(1-q)vv' - v^2]\int_0^q \overline{v}_\omega^2}{v^4(1-q)^2} \\
&< \frac{v^2(1-q)v^2 - (2v^2 - v^2)\int_0^q \overline{v}_\omega^2}{v^4(1-q)^2} \\
&= \frac{1}{1-q}\left[1 - \frac{1}{(1-q)v^2}\int_0^q \overline{v}_\omega^2\right].
\end{aligned}$$

Therefore, at such $q = q_*(\omega)$ we have

$$\begin{aligned}
\frac{\partial}{\partial q}r^*(q,\omega) &= \frac{-1}{1-q^*} + \frac{1-q^*}{\Gamma_\omega^*(\overline{v}_\omega(q^*))-1}\cdot\frac{\partial}{\partial q}\left[\frac{\int_0^q \overline{v}_\omega^2}{\overline{v}_\omega(q)^2(1-q)}\right] \\
&< \frac{-1}{1-q^*} + \frac{1-q^*}{\Gamma_\omega^*(\overline{v}_\omega(q^*))-1}\left\{\frac{1}{1-q}\left[1 - \frac{1}{(1-q)v^2}\int_0^q \overline{v}_\omega^2\right]\right\} \\
&= \frac{-1}{1-q}\left[r^*(q,\omega)+2\right] \\
&= \frac{-2}{1-q} < 0,
\end{aligned}$$

where $r^*(q,\omega) = 0$ due to Lemma 13. The quantity ranking follows. *Q.E.D.*

E.2. Proof of Proposition 6

First, we formulate our optimality condition more precisely. Say a tuple $(\Pi_1,\ldots,\Pi_N,q) \in [\Delta(\mathbb{R}_+)]^N \times [0,1]$ is *worst-feasible* if q is a worst equilibrium for the seller given that price distribution Π_n is used for each group $n \in \{1,\ldots,N\}$. Say a tuple $(\Pi_1^*,\ldots,\Pi_N^*,q^*) \in [\Delta(\mathbb{R}_+)]^N \times [0,1]$ is *limit-worst-feasible (LWF)* if it is a limit of a sequence of worst-feasible pairs. Let

$$R_N^* := \sup_{(\Pi_1,\ldots,\Pi_N,q)\text{ worst-feasible}} \sum_{n=1}^N \lambda_n R_{n,q}(\Pi_n)$$

denote the seller's optimal value. and say $(\Pi_1^*,\ldots,\Pi_N^*,q^*)$ is *optimal* if it is LWF with a witnessing sequence that has revenue converging to R_N^*.

Throughout this proof, for any notation used throughout our paper, and any $n \in \{1, \ldots, N\}$, let the same notation with a subscript n refer to the corresponding object for group n.

We begin our argument by noting that several results proven for our main model also apply directly to the setting with heterogeneity, with proofs applying verbatim or nearly verbatim. In particular, Lemma 3, Lemma 4, Lemma 5, Lemma 7, Lemma 8, Lemma 9, and the first sentence of Lemma 2 apply separately to each group; and the remainder of Lemma 2 (pertaining to the best-case and worst-case equilibrium quantities) applies with identical proof. Moreover, following essentially identically the proof of Proposition 1 tells us $(\Pi_1^*, \ldots, \Pi_N^*, q^*)$ is optimal if and only if it solves program (P_N^*), and that such an optimum exists and yields strictly positive revenue. In addition, various claims proven in the proof of Theorem 1 adapt to the present setting with essentially identical proofs. First, Claim 1 and Claim 4 adapt immediately to each group separately. Next, the analogues of Claim 2 and Claim 5, in which we replace $q \mapsto D_q(\Pi)$ with the function $q \mapsto \sum_{n=1}^{N} \lambda_n D_{n,q}(\Pi_n)$ (and for Claim 5 replace the modified price distribution with a modified profile of N price distributions), adapt immediately. In what follows, whenever we reference any of these results, we mean to apply these analogues.

Now, fix some optimum $(\Pi_1^*, \ldots, \Pi_N^*, q^*)$ for program (P_N^*). It will be convenient to work with the "partial demand" functions $\mathcal{D}_n : [0,1] \to [0,1]$, for $n \in \{0, \ldots, N\}$, given by

$$\mathcal{D}_n(q) := \sum_{m=1}^{n} \lambda_m D_{m,q}(\Pi_m^*).$$

The following quantitative result tells us when mass can be swapped (and how much relative mass needs to be swapped) between two different locations (in "quantity space") for two different price distributions, while preserving the demand constraints and raising expected revenue.

Claim 6. *Suppose $0 \leq q_0 < q_1 \leq q^*$ and $n_0, n_1 \in \{1, \ldots, N\}$ have $n_0 < n_1$. Let $v_0 := \overline{v}_{n_0}$ and $v_1 := \overline{v}_{n_1}$, and take $\gamma \in \mathbb{R}_+$. Define $\Delta D : [0,1] \to \mathbb{R}$ and*

$\Delta R \in \mathbb{R}$ *by letting*[55]

$$\begin{aligned}
\Delta D(q) &:= \gamma \left[D_{n_0,q}\left(v_0(q_0)\right) - D_{n_0,q}\left(v_0(q_1)\right)\right] + \left[D_{n_1,q}\left(v_1(q_1)\right) - D_{n_1,q}\left(v_1(q_0)\right)\right] \\
\Delta R &:= \gamma \left[R_{n_0,q^*}\left(v_0(q_0)\right) - R_{n_0,q^*}\left(v_0(q_1)\right)\right] + \left[R_{n_1,q^*}\left(v_1(q_1)\right) - R_{n_1,q^*}\left(v_1(q_0)\right)\right].
\end{aligned}$$

Then:

- *(i) Every* $q \in [0, q_0]$ *has* $\Delta D(q) = 0$.
- *(ii) If* $\gamma \geq \frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}$, *then every* $q \in [q_1, 1]$ *has* $\Delta D(q) \geq 0$.
- *(iii) If* $q \in (q_0, q_1)$ *has* $\gamma \geq \frac{v_0(q)}{v_1(q)} \cdot \frac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)}$, *then* $\Delta D(q) \geq 0$.
- *(iv) If* $v_0(q_1) \leq p^M_{n_0}(q^*)$ *and* $\gamma \leq \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)}$, *then* $\Delta R > 0$.

Proof. The first point follows trivially because each of the four demands in the definition of $\Delta D(q)$ is zero if $q \in [0, q_0]$.

Toward the second point, suppose γ satisfies the given inequality and $q \in [q_1, 1]$. That $D_{n_0,q}\left(v_0(q_0)\right) \geq D_{n_0,q}\left(v_0(q_1)\right)$ implies

$$\begin{aligned}
\Delta D(q) &\geq \frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[D_{n_0,q}\left(v_0(q_0)\right) - D_{n_0,q}\left(v_0(q_1)\right)\right] \\
&\quad + \left[D_{n_1,q}\left(v_1(q_1)\right) - D_{n_1,q}\left(v_1(q_0)\right)\right] \\
&= \frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left\{\left[1 - \frac{v_0(q_0)}{v_0(q)}\right] - \left[1 - \frac{v_0(q_1)}{v_0(q)}\right]\right\} \\
&\quad + \left\{\left[1 - \frac{v_1(q_1)}{v_1(q)}\right] - \left[1 - \frac{v_1(q_0)}{v_1(q)}\right]\right\} \\
&= \frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[-\frac{v_0(q_0)}{v_0(q)} + \frac{v_0(q_1)}{v_0(q)}\right] + \left[-\frac{v_1(q_1)}{v_1(q)} + \frac{v_1(q_0)}{v_1(q)}\right] \\
&= \frac{v_0(q_1)}{v_1(q_1)v_1(q)} \left[v_1(q_1) - v_1(q_0)\right] \left[\frac{v_1(q)}{v_0(q)} - \frac{v_1(q_1)}{v_0(q_1)}\right] \\
&\geq 0.
\end{aligned}$$

For the third point, suppose γ satisfies the given inequality for some given

[55] These respectively correspond to the marginal changes to quantity demanded (at anticipated quantity q) and to revenue (at quantity q^*) when we move a unit of group n_0 prices from $v_0(q_1)$ to $v_0(q_0)$ and move γ units of group n_1 prices from $v_1(q_0)$ to $v_1(q_1)$.

$q \in (q_0, q_1)$. That $D_{n_0,q}(v_0(q_0)) \geq D_{n_0,q}(v_0(q_1))$ implies

$$\begin{aligned}
\Delta D(q) &\geq \tfrac{v_0(q)}{v_1(q)} \cdot \tfrac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)} \left[D_{n_0,q}(v_0(q_0)) - D_{n_0,q}(v_0(q_1))\right] \\
&\quad + \left[D_{n_1,q}(v_1(q_1)) - D_{n_1,q}(v_1(q_0))\right] \\
&= \tfrac{v_0(q)}{v_1(q)} \cdot \tfrac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)} D_{n_0,q}(v_0(q_0)) - D_{n_1,q}(v_1(q_0)) \\
&= \tfrac{v_0(q)}{v_1(q)} \cdot \tfrac{v_1(q)-v_1(q_0)}{v_0(q)-v_0(q_0)} \left[1 - \tfrac{v_0(q_0)}{v_0(q)}\right] - \left[1 - \tfrac{v_1(q_0)}{v_1(q)}\right] \\
&= 0.
\end{aligned}$$

Finally, for the fourth point, suppose $v_0(q_1) \leq p^M_{n_0}(q^*)$ and γ satisfies the given inequality. That $v_0(q_1) \leq p^M_{n_0}(q^*)$ implies that $q_1 < q^*$ and (given Assumption 3) that $R_{n_0,q^*}(v_0(q_0)) < R_{n_0,q^*}(v_0(q_1))$. Therefore,

$$\begin{aligned}
\Delta R &\geq \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[R_{n_0,q^*}(v_0(q_0)) - R_{n_0,q^*}(v_0(q_1))\right] \\
&\quad + \left[R_{n_1,q^*}(v_1(q_1)) - R_{n_1,q^*}(v_1(q_0))\right] \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left\{v_0(q_0)\left[1 - \tfrac{v_0(q_0)}{v_0(q^*)}\right] - v_0(q_1)\left[1 - \tfrac{v_0(q_1)}{v_0(q^*)}\right]\right\} \\
&\quad + \left\{v_1(q_1)\left[1 - \tfrac{v_1(q_1)}{v_1(q^*)}\right] - v_1(q_0)\left[1 - \tfrac{v_1(q_0)}{v_1(q^*)}\right]\right\} \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[v_0(q_0) - v_0(q_1)\right] + \left[v_1(q_1) - v_1(q_0)\right] \\
&\quad + \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[-\tfrac{v_0(q_0)^2}{v_0(q^*)} + \tfrac{v_0(q_1)^2}{v_0(q^*)}\right] + \left[-\tfrac{v_1(q_1)^2}{v_1(q^*)} + \tfrac{v_1(q_0)^2}{v_1(q^*)}\right] \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_0(q_1)-v_0(q_0)} \left[-\tfrac{v_0(q_0)^2}{v_0(q^*)} + \tfrac{v_0(q_1)^2}{v_0(q^*)}\right] + \left[-\tfrac{v_1(q_1)^2}{v_1(q^*)} + \tfrac{v_1(q_0)^2}{v_1(q^*)}\right] \\
&= \left[v_1(q_1) - v_1(q_0)\right]\left[\tfrac{v_0(q_0)+v_0(q_1)}{v_0(q^*)} - \tfrac{v_1(q_1)+v_1(q_0)}{v_1(q^*)}\right] \\
&= \tfrac{v_1(q_1)-v_1(q_0)}{v_1(q^*)} \left\{v_0(q_0)\left[\tfrac{v_1(q^*)}{v_0(q^*)} - \tfrac{v_1(q_0)}{v_0(q_0)}\right] + v_0(q_1)\left[\tfrac{v_1(q^*)}{v_0(q^*)} - \tfrac{v_1(q_1)}{v_0(q_1)}\right]\right\} \\
&> 0,
\end{aligned}$$

where strictness in the last inequality follows from $0 < q_1 < q^*$. *Q.E.D.*

The following claim establishes that gaps (in "quantity space") in the price distribution are always bookended by atoms.

Claim 7. *Suppose* $0 \leq q_0 < q_1 < q^*$ *and* $[q_0, q_1] \cap \operatorname{supp} \overline{\Pi} = \{q_0, q_1\}$, *where* $\overline{\Pi} := \sum_{n=1}^{N} \lambda_n \Pi^*_n \circ \overline{v}_n$. *Then,* $\overline{\Pi}$ *has mass points at both* q_0 *and* q_1.

Proof. We proceed in two cases. First, consider the case in which $\mathcal{D}_N(q_0) = q_0$ [resp. $\mathcal{D}_N(q_1) = q_1$]. Note that Lemma 8 tells us the function $q \mapsto \mathcal{D}_N(q) - q$ is strictly concave on (q_0, q_1), hence (because it is nonnegative) strictly positive. Therefore, in this case, combining Claim 2 and Lemma 9 (exactly as in the proof of Claim 3) tells us $\overline{\Pi}$ has a mass point at q_0 [resp. q_1].

Second, consider the case in which $\mathcal{D}_N(q_0) > q_0$ [resp. $\mathcal{D}_N(q_1) > q_1$]. Assume for a contradiction that $\overline{\Pi}$ does not have a mass point at q_0 [resp. q_1]—or equivalently, that $\overline{\Pi}_n := \Pi_n^* \circ \overline{v}_n$ does not have a mass point there for any $n \in \{1, \ldots, N\}$. Lemma 5 implies $\mathcal{D}_N(q) > q$ for every q in some open interval Q around q_0 [resp. q_1]. Now, take some $n \in \{1, \ldots, N\}$ such that q_0 [resp. q_1] is in $\operatorname{supp} \overline{\Pi}_n$, which exists because it is in $\operatorname{supp} \overline{\Pi}$. Because it has a support point somewhere in Q where it does not have a mass point, $\overline{\Pi}_n$ must have multiple support points in Q. But then the mass in this interval can be replaced with its expectation. Doing so will preserve $\mathcal{D}_N(q)$ for $q \in (0, q^*) \setminus Q$, and will strictly improve the objective by Lemma 7. Moreover, Claim 5 tells us an appropriate proper weighted average of $\overline{\Pi}_n$ and the modified distribution also satisfies $\mathcal{D}_N(q) \geq q$ for $q \in Q$, in contradiction to the optimality of $(\Pi_n)_{n=1}^N$ in program (P_N^*). *Q.E.D.*

The following claim, together with (an appropriate generalization of) Theorem 1, is the heart of the proposition.

Claim 8. *Every* $n \in \{1, \ldots, N\}$ *has* $\Pi_n^*(p_n^M(q^*)) = 1$ *and, if* $n < N$, *has*

$$\max \operatorname{supp}(\Pi_n^* \circ \overline{v}_n) \leq \min \operatorname{supp}(\Pi_{n+1}^* \circ \overline{v}_{n+1}).$$

Proof. First, observe $\Pi_n^*(p_n^M(q^*)) = 1$ for every $n \in \{1, \ldots, N\}$. Indeed, if not, then applying Claim 1 for some group n with $\Pi_n^*(p_n^M(q^*)) < 1$ would violate optimality.

Letting $\overline{\Pi}_n := \Pi_n^* \circ \overline{v}_n \in \Delta[0, q^*]$ for each $n \in \{1, \ldots, N\}$, we now know that $(\overline{\Pi}_n)_{n=1}^N$ is an optimal solution to the program

$$\max_{(\hat{\Pi}_n)_n \in (\Delta[0,q^*])^N} \sum_{n=1}^N \lambda_n R_{n,q^*}(\hat{\Pi}_n \circ \underline{q}_n) \tag{$\overline{\mathrm{P}}_N^*$}$$

$$\text{subject to } \sum_{n=1}^N \lambda_n D_{n,\hat{q}}(\hat{\Pi}_n \circ \underline{q}_n) \geq \hat{q} \quad \forall \hat{q} \in (0, q^*).$$

In what follows, let us say (n_0, n_1, q_0, q_1) is a *mismatch* if $n_0, n_1 \in \{1, \ldots, N\}$ have $n_0 < n_1$, and $q_1 \in \operatorname{supp} \overline{\Pi}_{n_0}$ and $q_0 \in \operatorname{supp} \overline{\Pi}_{n_1}$ have $q_0 < q_1$. It remains to show that no mismatch exists.

Now, given $n_0, n_1 \in \{1, \ldots, N\}$ with $n_0 < n_1$, let us argue no $\hat{q} \in (0, q^*) \cap \left(\operatorname{supp} \overline{\Pi}_{n_0}\right) \cap \left(\operatorname{supp} \overline{\Pi}_{n_1}\right)$ is such that $\Pi_{n_0}^*$ has mass at $\hat{q}^{++}$ or $\Pi_{n_1}^*$ has mass at $\hat{q}^{--}$. Assume for a contradiction that such $\hat{q}$ does exist; we will show this assumption contradicts optimality in ($\overline{\mathrm{P}}_N^*$). To that end, note that the externality ranking implies $\frac{v_1(\hat{q})}{v_0(\hat{q})} > 1$, so that some $\gamma \in \left(\frac{v_0(\hat{q})v_1'(\hat{q})}{v_1(\hat{q})v_0'(\hat{q})}, \frac{v_1'(\hat{q})}{v_0'(\hat{q})}\right)$ exists. Because v_0, v_1 are continuously differentiable, and strictly positive on $(0, q^*)$ with strictly positive derivative there, some neighborhood of $\hat{q}$ must exist in $(0, q^*)$ such that any $q_0 < q_1$ in the neighborhood have

$$\tfrac{v_0(q_1)}{v_1(q_1)} \cdot \tfrac{v_1(q_1) - v_1(q_0)}{v_0(q_1) - v_0(q_0)} < \gamma < \tfrac{v_1(q_1) - v_1(q_0)}{v_0(q_1) - v_0(q_0)}.$$

By definition of $\hat{q}$, there exist some probability measures $\hat{\Pi}_0, \hat{\Pi}_1$ on this neighborhood such that some strictly positive multiple of $\hat{\Pi}_0$ is a submeasure of $\overline{\Pi}_{n_0}$, some strictly positive multiple of $\hat{\Pi}_1$ is a submeasure of $\overline{\Pi}_{n_1}$, and $\max \operatorname{supp} \hat{\Pi}_1 < \min \operatorname{supp} \hat{\Pi}_0$.[56] So let $\varepsilon > 0$ be small enough to ensure $\frac{\varepsilon\gamma}{\lambda_{n_0}} \hat{\Pi}_0$ is a submeasure of $\overline{\Pi}_{n_0}$ and $\frac{\varepsilon}{\lambda_{n_1}} \hat{\Pi}_1$ is a submeasure of $\overline{\Pi}_{n_1}$. We can then replace $\overline{\Pi}_{n_0}$ with $\overline{\Pi}_{n_0} + \frac{\varepsilon\gamma}{\lambda_{n_0}} \left(\hat{\Pi}_1 - \hat{\Pi}_0\right)$ and replace $\overline{\Pi}_{n_1}$ with $\overline{\Pi}_{n_1} + \frac{\varepsilon}{\lambda_{n_1}} \left(\hat{\Pi}_0 - \hat{\Pi}_1\right)$; Claim 6 ensures that this modified tuple of distributions remains feasible in ($\overline{\mathrm{P}}_N^*$) and strictly improves the objective—the desired contradiction.[57]

[56] Recall, a submeasure of a measure is any measure that assigns a weakly lower mass to any event. In the language of cumulative distribution functions, $\tilde{\Gamma}$ is a submeasure of Γ if $\Gamma - \tilde{\Gamma}$ is weakly increasing.

[57] Here and elsewhere in this proof, we apply Claim 6 when mass is moved from high to

Next, we argue that there cannot exist a mismatch (n_0, n_1, q_0, q_1) with $[q_0, q_1] \cap \operatorname{supp} \overline{\Pi} = \{q_0, q_1\}$, where $\overline{\Pi}$ is as defined in Claim 7. Indeed, assume otherwise for a contradiction, with the witnessing n_0, n_1 chosen to maximize $n_1 - n_0 > 0$. Let us now argue $\overline{\Pi}_{n_1}$ has a mass point at q_0, and $\overline{\Pi}_{n_0}$ has a mass point at q_1. To that end, note Claim 7 tells us $\overline{\Pi}_{\tilde{n}_1}$ [resp. $\overline{\Pi}_{\tilde{n}_0}$] has a mass point at q_0 [resp. q_1] for some $\tilde{n}_1 \in \{1, \ldots, N\}$ [resp. $\tilde{n}_0 \in \{1, \ldots, N\}$], and maximality of $n_1 - n_0 > 0$ tells us $\tilde{n}_1 \leq n_1$ [resp. $\tilde{n}_0 \geq n_0$]. If $n_1 = \tilde{n}_1$ [resp. $n_0 = \tilde{n}_0$], then $\overline{\Pi}_{n_1}$ [resp. $\overline{\Pi}_{n_0}$] has a mass point at q_0 [resp. q_1] by definition. If $\tilde{n}_1 < n_1$ [resp. $\tilde{n}_0 > n_0$], then applying the previous paragraph to $(\tilde{n}_1, n_1)$ [resp. $(n_0, \tilde{n}_0)$] tells us $\Pi^*_{n_1}$ [resp. $\Pi^*_{n_0}$] cannot have mass at q_0^{--} [resp. q_1^{++}]; but $\overline{\Pi}_{n_1}$ [resp. $\overline{\Pi}_{n_0}$] also cannot have any mass in (q_0, q_1), and so having q_0 [resp. q_1] in its support implies it has a mass point there. Fixing any $\gamma \in \left(\frac{v_0(q_1)}{v_1(q_1)} \cdot \frac{v_1(q_1) - v_1(q_0)}{v_0(q_1) - v_0(q_0)}, \ \frac{v_1(q_1) - v_1(q_0)}{v_0(q_1) - v_0(q_0)} \right)$, we can then find some $\varepsilon > 0$ small enough that $\overline{\Pi}_{n_0}$ puts mass at least $\frac{\varepsilon \gamma}{\lambda_{n_0}}$ on q_1 and $\overline{\Pi}_{n_1}$ puts mass at least $\frac{\varepsilon}{\lambda_{n_1}}$ on q_0. We can then replace $\overline{\Pi}_{n_0}$ with $\overline{\Pi}_{n_0} + \frac{\varepsilon \gamma}{\lambda_{n_0}} \left(\mathbf{1}_{[q_0, \infty)} - \mathbf{1}_{[q_1, \infty)} \right)$ and replace $\overline{\Pi}_{n_1}$ with $\overline{\Pi}_{n_1} + \frac{\varepsilon}{\lambda_{n_1}} \left(\mathbf{1}_{[q_1, \infty)} - \mathbf{1}_{[q_0, \infty)} \right)$. Claim 6 ensures that this modified tuple of distributions strictly improves the objective in ($\overline{\mathrm{P}}^*_N$), and preserves the constraint that $\sum_{n=1}^{N} \lambda_n D_{n,\hat{q}}(\hat{\Pi}_n \circ \underline{q}_n) \geq \hat{q} \ \ \forall \hat{q} \in (0, q^*)$ for any $\hat{q} \in (0, q^*) \setminus (q_0, q_1)$. Moreover, the nonnegative function $q \mapsto \mathcal{D}_N(q) - q$ is strictly concave on (q_0, q_1) by Lemma 8, and so strictly positive there. Hence, Claim 5 then tells us this modification is feasible in ($\overline{\mathrm{P}}^*_N$)—contradicting optimality—if ε is sufficiently small.

Now, let us show no mismatch (n_0, n_1, q_0, q_1) exists with $q_0, q_1 > 0$. Indeed, assume otherwise for a contradiction. For any n_0, n_1 and any $\tilde{q} \in (0, 1)$, let

$$Q(n_0, n_1, \tilde{q}) := \left\{ (q_0, q_1) \in [\tilde{q}, 1]^2 : \ (n_0, n_1, q_0, q_1) \text{ is a mismatch} \right\}.$$

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low quantities for $\overline{\Pi}_0$ and proportionately moved in the opposite direction for $\overline{\Pi}_1$, even if the mass being moved does not correspond to an atom. In all of these cases, we apply the claim pointwise, and the inequalities in Claim 6's statement imply corresponding rankings for the modified price distributions via integration.

The contradiction hypothesis tells us that that sufficiently small $\tilde{q} \in (0,1)$ has

$$Q(\tilde{q}) := \bigcup_{n_0,n_1\in\{1,\ldots,N\}:\ n_0<n_1} Q(n_0,n_1,\tilde{q}) \neq \emptyset.$$

Fixing such a $\tilde{q}$, let $(q_0^k, q_1^k)_{k=1}^\infty$ be some sequence from $Q(\tilde{q})$ such that $q_1^k - q_0^k$ converges as $k \to \infty$ to $\inf\{q_1 - q_0 :\ (q_0,q_1) \in Q(\tilde{q})\}$. Dropping to a subsequence, we may assume that the sequences $(q_0^k)_{k=1}^\infty$ and $(q_1^k)_{k=1}^\infty$ are both monotone (hence convergent), and that some $n_0, n_1 \in \{1,\ldots,N\}$ with $n_0 < n_1$ have $\left\{(q_0^k, q_1^k)\right\}_{k=1}^\infty \subseteq Q(n_0,n_1,\tilde{q})$. We now derive a contradiction in each of two cases. First, if $\lim_{k\to\infty}\left(q_1^k - q_0^k\right) = 0$, then n_0 and n_1, together with $\hat{q} := \lim_{k\to\infty} q_0 = \lim_{k\to\infty} q_1$, contradicts the claim of the paragraph before the previous one. Second, if $\lim_{k\to\infty}\left(q_1^k - q_0^k\right) > 0$, then $(q_0,q_1) := \lim_{k\to\infty}(q_0^k, q_1^k)$ belongs to $Q(n_0,n_1,\tilde{q})$ and satisfies $q_1 - q_0 = \min\{\hat{q}_1 - \hat{q}_0 :\ (\hat{q}_0,\hat{q}_1) \in Q(\tilde{q})\}$. But this minimality tells us no $n \in \{1,\ldots,N\}$ can have any $q \in (q_0,q_1) \cap \operatorname{supp} \overline{\Pi}_n$: if $n < n_1$ we would have $(q_0,q) \in Q(n,n_1,\tilde{q})$, and if $n > n_0$ we would have $(q,q_1) \in Q(n_0,n,\tilde{q})$. But then (n_0,n_1,q_0,q_1) contradicts the claim of the previous paragraph.

All that remains is to show we cannot have a mismatch of the form $(n_0,n_1,0,q_1)$. Assume otherwise, for a contradiction. If some $q_0 \in (0,q_1) \cap \operatorname{supp} \overline{\Pi}_{n_1}$ exists, then (n_0,n_1,q_0,q_1) is a mismatch with $q_0,q_1 > 0$—a contradiction. So we can now focus on the case that no such q_0 exists, implying $\overline{\Pi}_{n_1}$ has a mass point at zero. Therefore, letting $\varepsilon := \min\left\{\lambda_{n_0}, \lambda_{n_1}\overline{\Pi}_{n_1}(0)\right\} > 0$, we can then replace $\overline{\Pi}_{n_0}$ with $\overline{\Pi}_{n_0} + \frac{\varepsilon}{\lambda_{n_0}}\left(\mathbf{1}_{[0,\infty)} - \overline{\Pi}_{n_0}\right)$ and replace $\overline{\Pi}_{n_1}$ with $\overline{\Pi}_{n_1} + \frac{\varepsilon}{\lambda_{n_1}}\left(\overline{\Pi}_{n_0} - \mathbf{1}_{[0,\infty)}\right)$; Claim 6 ensures that this modified tuple of distributions remains feasible in ($\overline{\mathrm{P}}_N^*$) and strictly improves the objective—the desired contradiction. *Q.E.D.*

Claim 8 in particular tells us for each $n \in \{1,\ldots,N\}$ that we can view

$$\overline{\Pi}_n := \Pi_n^* \circ \overline{v}_n$$

as an element of $\Delta[0,q^*]$. Let $q_n^0 := \min \operatorname{supp} \overline{\Pi}_n$ and $q_n^1 := \max \operatorname{supp} \overline{\Pi}_n$, and observe that $p_n^* := \max \operatorname{supp} \Pi_n^* = \overline{v}_n(q_n^1)$ by construction.

The next claim adapts arguments from the greediness proof in Theorem 1 and its Corollary 1 to the present setting.

Claim 9. *Every* $n \in \{1, \ldots, N\}$ *has* $\operatorname{supp} \Pi_n^* = [\overline{v}_n(q_n^0), p_n^*]$ *and*

$$\mathcal{D}_N(q) = \mathcal{D}_n(q) = q, \ \forall q \in (q_n^0, q_n^1).$$

Moreover, Π_N^* *has a mass point at* p_N^*.

Proof. First observe that $\mathcal{D}_n$ and $\mathcal{D}_N$ agree on $[0, q_n^1]$ by Claim 8. The rest of the claim holds vacuously if $q_n^0 = q_n^1$, so focus now on the nontrivial case in which $q_n^0 < q_n^1$.

To show the result, we follow the proof of Theorem 1. We have already noted how Claim 1, Claim 2, Claim 4, and Claim 5 adapt. Moreover, an analogue of Claim 3 goes through—with the set $\left\{q \in \left(0, \underline{q}(p^*)\right) : \ D_q(\Pi) > q\right\}$ being replaced by $\{q \in (q_n^0, q_n^1) : \ \mathcal{D}_N(q) > q\}$ for some group n—given that Claim 8 says $\overline{\Pi}_m$ has no mass on (q_n^0, q_n^1) for $m \neq n$.

Applying the above five (modified) claims analogously to in the proof of Theorem 1, we learn that $(\Pi_1^*, \ldots, \Pi_N^*, q^*)$ could be modified to maintain feasibility in program (P_N^*) and strictly improve the objective if $\mathcal{D}_N(q) > q$ held for some $q \in (q_n^0, q_n^1)$. Thus, optimality tells us $\mathcal{D}_N(q) = q$ for every $q \in (q_n^0, q_n^1)$.

Moreover, adapting the proof of Lemma 10, we learn that $\{\overline{\Pi}_m\}_{m=1}^N$ cannot all be constant over any subinterval of (q_n^0, q_n^1). Thus (given Claim 8) $\overline{\Pi}_n$ is strictly increasing on this interval, meaning its support is equal to $[q_n^0, q_n^1]$ exactly.

Finally, following the argument for a mass point at the top of the support in the proof of Theorem 1 tells at least one of $\{\overline{\Pi}_n\}_{n=1}^N$ has a mass point at $\max \operatorname{supp} \left[\sum_{n=1,\ldots,N} \overline{\Pi}_n\right]$, which is equal to q_N^1 by Claim 8. But again by Claim 8, either $q_n^1 < q_N^1$ for every $n \in \{1, \ldots, N-1\}$ or $\overline{\Pi}_N$ is degenerate on q_N^1; in either case, $\overline{\Pi}_N$ has a mass point at q_N^1. Thus, Π_N^* has a mass point at p_N^*. *Q.E.D.*

With these claims in hand, we are now prepared to prove the proposition. Toward the support ranking, observe any $n \in \{1, \ldots, N-1\}$ has

$$\min \operatorname{supp} \Pi^*_{n+1} = \overline{v}_{n+1}(q^0_{n+1}) \geq \overline{v}_n(q^0_{n+1}) \geq \overline{v}_n(q^1_n) = \max \operatorname{supp} \Pi^*_n, \qquad (8)$$

where the first inequality follows from $\overline{v}_{n+1}$ having stronger externalities than $\overline{v}_n$, and the second inequality follows from Claim 8. Toward the strict ranking, it suffices to see we cannot have $q^0_{n+1} = 0 < q^1_{n+1}$. Indeed, this would allow us to show that $\min \operatorname{supp} \Pi^*_{n+1} > \max \operatorname{supp} \Pi^*_n$: If $q^0_{n+1} > 0$, then $\overline{v}_{n+1}$ having stronger externalities than $\overline{v}_n$ would mean that in fact the first inequality in (8) holds strictly; and if $q^1_{n+1} = 0$, then Claim 8 would tell us $p^*_n = p^*_{n+1} = 0$. So let us assume $q^0_{n+1} = 0 < q^1_{n+1}$ for a contradiction. Claim 8 then tells us Π^*_n is degenerate on 0, so that every $q \in [0, 1]$ has $\mathcal{D}_N(q) \geq \mathcal{D}_N(0) \geq \lambda_n > 0$, whereas Claim 9 tells us sufficiently small $q \in (0, \lambda_n)$ has $\mathcal{D}_N(q) = q$—a contradiction.

It remains to show that $(\Pi^*_n)^N_{n=1}$ are residual greedy up to $(p^*_n)^N_{n=1}$. In light of Claim 9, we need only show for each $n \in \{1, \ldots, N\}$ that

$$q_{n-1} := \max \{q \in [0, 1] : \; \mathcal{D}_{n-1}(q) = q\}$$

is well-defined, that $q^0_n = q^1_n$ if $q^1_n \leq q_{n-1}$, and that $q^0_n = q_{n-1}$ if $q^1_n > q_{n-1}$. All three claims are immediate if $n = 1$, because $q_0 = 0$ by definition, $0 \leq q^0_1 \leq q^1_1$, and feasibility in (P^*_N) requires (given that $q^0_1 = \min_{m=1,\ldots,N} q^0_m$ by Claim 8) that $q^0_1 = 0$. So let us focus on the nontrivial case in which $n > 1$. Observe now that $q \mapsto \mathcal{D}_{n-1}(q) - q$ is nonnegative at q^1_{n-1} (given feasibility in (P^*_N)), nonpositive at 1, and continuous (by Lemma 5) and strictly concave (by Lemma 8) on $\left[q^1_{n-1}, 1\right]$. The function either crosses zero from above exactly once in $(q^1_{n-1}, 1]$, or it is strictly decreasing on $[q^1_{n-1}, 1]$. In either case, q_{n-1} is well-defined, and $\mapsto \mathcal{D}_{n-1}(q) - q$ is strictly positive on (q^1_{n-1}, q_{n-1}) and strictly negative on $(q_{n-1}, 1]$.

We now need to show, for $n > 1$, that $q^0_n = q^1_n$ if $q^1_n \leq q_{n-1}$, and that $q^0_n = q_{n-1}$ if $q^1_n > q_{n-1}$. Toward the first assertion, suppose $q^1_n \leq q_{n-1}$. Then any $q \in (q^1_{n-1}, q^1_n)$ has $\mathcal{D}_N(q) \geq \mathcal{D}_{n-1}(q) > q$. Meanwhile, Claim 9 says any

$q \in (q_n^0, q_n^1)$ has $\mathcal{D}_N(q) = q$. Hence, (q_n^0, q_n^1) and (q_{n-1}^1, q_n^1) are disjoint. Then, because Claim 8 tells us $q_{n-1}^1 \leq q_n^0 \leq q_n^1$, it follows that $q_n^0 = q_n^1$ as desired. Finally, toward the second assertion, suppose $q_n^1 > q_{n-1}$. We want to show $q_n^0 = q_{n-1}$. If any $q \in (q_{n-1}, q_n^0)$ existed, it would (given Claim 8) have $\mathcal{D}_N(q) = \mathcal{D}_{n-1}(q) < q$, in violation of feasibility in (P_N^*). And if any $q \in (q_n^0, q_{n-1})$ existed, it would have $\mathcal{D}_N(q) \geq \mathcal{D}_{n-1}(q) > q$, in contradiction to Claim 9. Thus, no q lies strictly between q_n^0 and q_{n-1}, meaning they coincide. The proposition follows. *Q.E.D.*