Buying from a Group

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Motivation

Land developer interested in a large plot

Different parcels owned by different landholders

Acquire the whole plot or nothing

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- Payment proportional to land endowment

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Questions:

1. Which trading rules are optimal?
2. Are simple mechanisms optimal?
3. Which sellers are influential in optimal mechanism?

Model

ENVIRONMENT

Buyer and a group $i \in N = \{1, \dots, N\}$ of multiple sellers

Allocation: probability $\mathtt{x} \in [0, 1]$ of trade

Transfers: $(\mathtt{m}_i)_{i \in N} \in \mathbb{R}^N$ paid by the buyer to respective sellers

- Mechanisms *required* to satisfy $\mathtt{m}_i = \sigma_i \sum_j \mathtt{m}_j$
- Shares $\sigma_i > 0$ exogenous, with $\sum_j \sigma_j = 1$
- Let $\mathtt{m} := \sum_j \mathtt{m}_j$, scalar

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Buyer's profit: $b\mathtt{x} - \mathtt{m}$

Each seller i has payoff $\mathtt{m}_i - \sigma_i \boldsymbol{\theta}_i \mathtt{x}$, where $\boldsymbol{\theta}_i$ is private

Interpreting the model

Land development

- Seller i has land share $\sigma_i > 0$ and per-unit value $\boldsymbol{\theta}_i$

$$u_i = \mathtt{m}_i - \sigma_i \boldsymbol{\theta}_i \mathtt{x}$$

- Notice $u_i \propto \mathtt{m} - \boldsymbol{\theta}_i \mathtt{x}$

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Other examples

- Commodity cartels (σ_i = production share)
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Selling to a group

- Maintenance for homeowners' association
- Committee purchasing by organization (collective funds)

DISTRIBUTIONAL ASSUMPTIONS

Types $\boldsymbol{\theta}_i \in \Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$ satisfy:

- Independence
- $\underline{\theta}_i < b < \bar{\theta}_i$
- CDF F_i admits a continuous, strictly positive density f_i
- Virtual value $\varphi_i(\theta_i) := \theta_i + \frac{F_i}{f_i}(\theta_i)$ strictly increasing

$\implies$ $\boldsymbol{\varphi}_i := \varphi_i(\boldsymbol{\theta}_i)$ is also atomless with convex support

Related (Theoretical) Literature

Mechanisms for public goods

- e.g., d'Aspremont Gérard-Varet 1979, Güth Hellwig 1986, Mailath Postlewaite 1990

Voting mechanisms without transfers

- e.g., Rae 1969, Azrieli Kim 2014

Property rights in mechanism design

- e.g., Myerson Satterthwaite 1983, Cramton Gibbons Klemperer 1987

Posted-price mechanisms

- e.g., Riley Zeckhauser 1983, Hart Nisan 2017

Redistribution in mechanism design

- e.g., Mirrlees 1971, Dworczak Kominers Akbarpour 2021

Mechanisms

An **allocation rule** is measurable $x : \Theta \to [0, 1]$

A **transfer rule** is bounded measurable $m : \Theta \to \mathbb{R}$

A **(direct) mechanism** is a pair (x, m) of both

Mechanism is…

- **incentive compatible (IC)** if

$$\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} \mathbb{E}\left[m(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - \theta_i x(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\right] \forall i \in N, \ \forall \theta_i \in \Theta_i$$

- **individually rational (IR)** if

$$\mathbb{E}\left[m(\theta_i, \boldsymbol{\theta}_{-i}) - \theta_i x(\theta_i, \boldsymbol{\theta}_{-i})\right] \geq 0 \ \forall i \in N, \ \forall \theta_i \in \Theta_i$$

[Veto rights]

- **optimal** if it maximizes profit $\mathbb{E}[bx(\boldsymbol{\theta}) - m(\boldsymbol{\theta})]$ among IC, IR

Implementability

WHICH ALLOCATION RULES ARE IMPLEMENTABLE?

Which x admit m such that (x, m) is IC?

Lemma: Given allocation rule x, the following are equivalent:

1. Some m makes (x, m) IC.
2. x is interim monotone, i.e., X_i decreasing $\forall i$.

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- Can use $m(\theta) :=$ constant $+ \sum_i M_i(\theta_i)$
  - Translating interim transfer doesn't affect IC

Revenue equivalence for trade with a group

- Interim transfer rule for agent i pinned down by payment formula, up to constant
- $\mathbb{E}M_i(\boldsymbol{\theta}_i) = \mathbb{E}m(\boldsymbol{\theta}) = \mathbb{E}M_j(\boldsymbol{\theta}_j)$ by iterated expectations
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Lemma: If x is interim monotone, then buyer's optimal value among IC, IR mechanisms with allocation rule x is

$$\min_{i \in N} \mathbb{E}\left[x(\boldsymbol{\theta})(b - \varphi_i)\right]$$

Proof idea:

- Make IR bind for someone

Optimal mechanisms

Optimal allocation rule

Given $\omega \in \Delta N$, define the allocation rule x_ω by

$$x_\omega(\theta) := \mathbb{1}_{\omega \cdot \varphi(\theta) \leq b}$$

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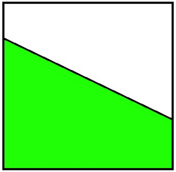
The "unique" optimal allocation rule is x_ω for some ω.



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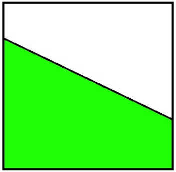

Can say more

- Which ω is optimal?
- (Of course:) What interim transfers go with it?

Optimal allocation rule: proof ideas

Solve

$$\max_{x} \min_{i} \mathbb{E}\left[x(\boldsymbol{\theta})\,(b - \varphi_i)\right]$$

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$$\max_{x} \quad \min_{i} \quad \mathbb{E}\left[x(\boldsymbol{\theta})\left(b - \varphi_i\right)\right]$$

Seek maximin strategies for this two-player zero-sum game:

- Maximizer chooses allocation rule $x/_{\text{a.e.}}$
- Minimizer chooses $i \in N$—mixed strategy $\omega \in \Delta N$
- Payoff is $g(x, \omega) := \mathbb{E}\left[x(\boldsymbol{\theta})\left(b - \omega \cdot \boldsymbol{\varphi}\right)\right]$

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Features of this 2PZS game:

- The game is convex/compact/affine/continuous (weak*)
- Every ω admits a unique Max best response: x_ω

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So unique maximin strategy x exists, and:

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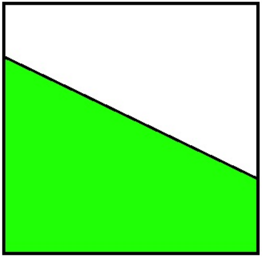
Moreover,

$$\begin{aligned} \omega \text{ played in NE} &\iff \omega \text{ best response to } x_\omega \\ &\iff \omega \text{ minimax} \end{aligned}$$

Optimal allocation rule

Theorem:
An interim-monotone allocation rule x^* is optimal iff $x^*(\boldsymbol{\theta}) = x_\omega(\boldsymbol{\theta})$ almost surely, for *unique* $\omega \in \Delta N$ satisfying these equivalent conditions:

- $\operatorname{supp} \omega \subseteq \arg\max_{i \in N} \mathbb{E}[\varphi_i \mid \omega \cdot \boldsymbol{\varphi} \leq b]$
- ω solves $\min_{\omega \in \Delta N} \mathbb{E}[(b - \omega \cdot \boldsymbol{\varphi})_+]$



An example

Say $N = 2$ and $\boldsymbol{\theta}_i \sim \mathcal{U}[0, 1]$

An optimal (indirect) mechanism:

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Say $N = 2$ and $\boldsymbol{\theta}_i \sim \mathcal{U}[0, 1]$

An optimal (indirect) mechanism:

- Bidders simultaneously bid $s_i \geq 0$
- Provisional price is equal to $p = p(s_1, s_2) = s_1 + s_2$
- Trade at price p if and only if $b \geq p$
- Equilibrium: bid = type

Posted prices

Defining posted prices

Posted prices: ubiquitous simple mechanism

- Exact optimality: Myerson 1981, Riley Zeckhauser 1983
- Approximate optimality: Hart Nisan 2017

With one agent, posted price has two features

- Transfer $\propto$ probability of trade
- Allocation step function: agent chooses whether to trade

How to generalize with multiple agents?

DEFINING POSTED PRICES

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How to generalize with multiple agents?

Definition:
Say (x, m) is a **(collective) posted price** if $m = px$ for some $p \in \mathbb{R}$.

Suboptimality of posted prices

Proposition:

No posted price mechanism is optimal.

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Interpretation: optimal mechanisms typically

- Rely "smoothly" on sellers' private information

Suboptimality of posted prices

Quantifying gains of richer pricing

Suppose $\boldsymbol{\theta}_i \sim \mathcal{U}[0,1]$, and b is low enough that $X_i(1) = 0$

Then $$\frac{\text{optimal value}}{\text{optimal value from posted price}} = \frac{(N+1)^N}{N!\, 2^N}$$

- If $N = 2$, then $= 1.125$
- If $N = 5$, then $= 2.025$
- If $N = 10$, then ≈ 6.98
- If $N = 25$, then ≈ 455

SUBOPTIMALITY OF POSTED PRICES: PROOF IDEA

Consider any IC collective posted price with price p

- $M_i(\hat{\theta}_i) - \theta_i X_i(\hat{\theta}_i) = (p - \theta_i) X_i(\hat{\theta}_i)$
- IC $\implies$ X_i constant below and above p

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Now, consider any optimal mechanism: x_ω

- If $\omega_i < 1$, then X_i continuous
- If $\omega_i > 0$, then X_i non-constant because

$$\omega \cdot \varphi(\underline{\theta}) < b < \omega \cdot \varphi(\bar{\theta})$$

- Can't have $\omega_i = 1$ because $j \neq i$ has

$$\mathbb{E}\left[\boldsymbol{\varphi}_j \mid \boldsymbol{\varphi}_i \leq b\right] = \mathbb{E}[\boldsymbol{\varphi}_j] = \bar{\theta}_j > b > \mathbb{E}\left[\boldsymbol{\varphi}_i \mid \boldsymbol{\varphi}_i \leq b\right]$$

The role of heterogeneity

Ranking weights

Optimal mechanism described by endogenous weights $(\omega_i)_i$

Weights determine who we "pay attention" to

Relationship between ω and seller characteristics?

Ranking weights

Definition: Let $\mathbf{y}_L$ and $\mathbf{y}_H$ be random variables with CDFs F_L and F_H. Say $\mathbf{y}_H$ is above $\mathbf{y}_L$ in the **reversed hazard-rate order** (denoted $\mathbf{y}_H \gtrsim \mathbf{y}_L$) if $\inf \operatorname{supp}(\mathbf{y}_L) \leq \inf \operatorname{supp}(\mathbf{y}_H)$ and $\frac{F_H}{F_L}$ is weakly increasing above $\inf \operatorname{supp}(\mathbf{y}_L)$

Interpretation: FOSD conditional on being below any cutoff

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Uniqueness $\implies$ enough to show $\tilde{\omega}$ with flipped (ω_i, ω_j) has

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So letting $\eta(y) := \mathbb{E}\left[\min\left\{0,\ y - b + \sum_{k \neq i,j} \omega_k \varphi_k\right\}\right]$, need

$$\mathbb{E}\eta\left(\omega_i \varphi_i + \omega_j \varphi_j\right) \leq \mathbb{E}\eta\left(\omega_j \varphi_i + \omega_i \varphi_j\right)$$

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Off-the-shelf stochastic ranking result—η increasing concave

Ranking weights & land shares

Giving higher weight to agents with higher value distributions

In principle, independent of land shares

- If $\boldsymbol{\theta}_i \sim \boldsymbol{\theta}_j$, then $\omega_i = \omega_j$ whatever σ_i and σ_j are.

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Natural relationship between σ_i and F_i depends on setting

- Farming vs. manufacturing? Higher $\sigma_i \rightsquigarrow$ higher ω_i.
- Small vs. medium farm? Higher $\sigma_i \rightsquigarrow$ lower ω_i.

An advertisement

See the paper for...

Dominant strategies

Ex-post participation

Beyond veto bargaining

Pre-market trade

The full Pareto frontier

Wrapping up

What we've seen

Model of buying from seller group with shared property rights

Proportional transfers don't hamper implementability

Optimally use weighted allocation rule—endogenous weights

Simple pricing leaves money on the table

Weights reflect heterogeneity: value ranking $\rightsquigarrow$ weight ranking

Thanks!