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Knowing the informed player's payoffs and simple play in repeated games [☆]

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ABSTRACT

We revisit the canonical model of repeated games with two patient players, observable actions, and one-sided incomplete information, and make the substantive assumption that the informed player's preference is state independent. We show the informed player can attain a payoff in equilibrium if and only if she can attain it in the simple class of equilibria first studied by Aumann, Maschler, and Stearns (1968), in which the initial stages are used only for revealing information, and no further information is revealed after the initial stages. This sufficiency result does not extend to the uninformed player's equilibrium payoff set.

How should an informed party behave when repeatedly interacting with others who are not informed? Because information may be revealed implicitly by a player's actions and used against the player, an informed player may sometimes wish to refrain from taking certain actions to avoid leakage of information. Alternatively, an informed player may wish to take certain actions to signal information. A rich literature, beginning with Aumann and Maschler (1966), has studied this question using the framework of repeated games with incomplete information.[1] In Aumann and Maschler's (1966) model, two patient players play a repeated zero-sum game in which only one player is informed about the state of the world. Our paper takes the non-zero-sum version of this framework (developed by Aumann et al., 1968), and adds the assumption that the informed player's payoffs are known.[2] Our substantive assumption is familiar from communication games that our model nests (e.g., Chakraborty and Harbaugh, 2010; Lipnowski and Ravid, 2020), and appropriate in many situations: for example, political parties want to win elections, dynasties want to remain in power, and companies want to continue to sell their goods to consumers.

Our main result, Proposition 1, shows that when the informed player's payoffs are known, any equilibrium payoff for the informed player can be attained in an equilibrium in which the initial stages are used only for revealing information, and no further informa-

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1 See the preface of Aumann and Maschler (1995) and the foreword in Mertens et al. (2015) for historiographical accounts.

2 Like Aumann et al. (1968), we assume the uninformed player's payoffs are state dependent. As discussed by Mertens (1987) and others, this assumption is justified whenever we can interpret payoffs as representing how players would feel if they were fully informed of the game's outcome, including the state, rather than how players feel during the game. This interpretation is reasonable in many scenarios, such as in principal-agent relationships (which often feature a principal who cares about the agent's information, but can learn this information only through the agent's actions), and environments in which players care about latent variables (e.g., long-run health outcomes).

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tion is revealed after the initial stages. Thus, we show that gradually leaking or signaling information has no effect on the informed player's equilibrium payoffs whenever her motives are transparent. Our main result simplifies the computation of the informed player's equilibrium payoffs compared to the general payoff case, which sometimes requires gradual communication of information.[3] In particular, our Corollary 1 enables one to obtain the informed player's equilibrium payoffs using the technique of quasiconvexification. We also demonstrate that even when the informed player's motives are transparent, gradual communication of information is sometimes needed to obtain equilibrium payoffs for the uninformed player.

We focus on environments in which players are patient and we maintain Aumann and Maschler's (1966) approach of modeling patient players directly; i.e., we assume that players care only about their long-run average payoffs.[4] An alternative approach for modeling such situations is to consider the patient limit of environments with impatient players—that is, players who care about their expected discounted payoffs but discount future payoffs by very little. In the appendix, we show conditions under which these patient limits include the set of payoffs identified in our paper.[5] Therefore, the set of equilibrium payoffs with patient players we characterize is also relevant in understanding equilibrium payoffs in discounted games.

The proof of our main result exploits the connection between games of pure communication and repeated games of incomplete information. Forges (2020) provides a survey that details the connections between these two classes of games. In particular, we appeal to results from Lipnowski and Ravid (2020) on cheap talk games in which the sender's payoffs are known to the repeated games studied by Aumann et al. (1968) and Hart (1985) while assuming that the informed player's payoffs are known.

1. Model

We study the two-player repeated game of one-sided incomplete information with undiscounted utility and observable actions from Aumann et al. (1968) and Hart (1985), and specialize the informed player's preference to be state independent. Formally, the game has two players: one *informed* (player 1) and one *uninformed* (player 2). The game begins with a realization of a payoff-relevant random state θ from a finite set Θ (with at least two elements) according to a full-support distribution $\mu_0 \in \Delta\Theta$.[6] Then, player 1 observes the realization of θ, and the players subsequently play the stage game infinitely many times. In each period $t \in \mathbb{N}$, each player $j \in \{1,2\}$ chooses an action from a finite set A_j (with at least two elements) simultaneously, and stage payoffs are given by $u_1 : A \to \mathbb{R}$ and $u_2 : A \times \Theta \to \mathbb{R}$ for players 1 and 2, respectively, where $A := A_1 \times A_2$. At the end of each period $t \in \mathbb{N}$, players observe the period's chosen action profile, $a^t = (a_1^t, a_2^t) \in A$, but not the resulting payoffs. The assumption that the players do not observe payoffs means that player 2 can learn about the state only from player 1's actions. We assume players have limit-of-means preferences over sequences of payoffs as formalized in the definition of equilibrium below.

Let $\sigma_1 : \mathcal{H} \times \Theta \to \Delta A_1$ and $\sigma_2 : \mathcal{H} \to \Delta A_2$ denote player 1 and 2's behavior strategies, respectively, where $\mathcal{H} := \bigcup_{t=0}^{\infty} A^t$ is the set of public histories. Let $\mathbb{E}_{\sigma,\mu}$ denote the expectation operator with respect to the unique probability measure on $\Omega := A^\infty \times \Theta$ induced by a strategy profile $\sigma = (\sigma_1, \sigma_2)$ and a belief $\mu \in \Delta\Theta$.[7] Define expectations of players 1 and 2's payoffs up to and including stage $t \in \mathbb{N}$, respectively, as

$$v_1^t(\sigma) := \mathbb{E}_{\sigma,\mu_0}\left[\frac{1}{t}\sum_{\tau=1}^{t} u_1(a^\tau)\right],\ v_2^t(\sigma) := \mathbb{E}_{\sigma,\mu_0}\left[\frac{1}{t}\sum_{\tau=1}^{t} u_2(a^\tau,\theta)\right]. \tag{1.1}$$

Following Aumann and Maschler (1968),[8] a strategy profile σ is an *equilibrium* if the following two conditions hold:

$$\liminf_{t\to\infty} v_1^t(\sigma) \geq \limsup_{t\to\infty} \sup_{\sigma_1'} v_1^t\left(\sigma_1', \sigma_2\right), \tag{1.2}$$

$$\liminf_{t\to\infty} v_2^t(\sigma) \geq \limsup_{t\to\infty} \sup_{\sigma_2'} v_2^t\left(\sigma_1, \sigma_2'\right). \tag{1.3}$$

The payoffs for the players associated with an equilibrium σ are their respective limit payoffs[9]:

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[3] See, for example, Forges (1984; 1994).

[4] See section 3.2.2 in Mailath and Samuleson (2006) for a discussion of various approaches of modeling patient players' preferences.

[5] Several other papers have obtained related results, but under different assumptions. For example, in repeated games with complete information, the set of equilibrium payoffs in the game with patient players and the patient limit of games with impatient players essentially coincide (Rubinstein, 1979; Fudenberg and Maskin, 1986; Aumann and Shapley, 1994). Cripps and Thomas (2003) introduce impatient players to the environment of Aumann et al. (1968) while assuming that the uninformed player's payoffs are known (i.e., the "known-own payoff" case) and show that, when the informed player is arbitrarily patient relative to the uniformed player, the characterization of the informed player's payoffs are essentially the same as in the game with patient players. On the other hand, Cripps and Thomas (2003) and Pęski (2008; 2014) also consider the equal discounting case (with known-own payoffs) and show that the set of equilibrium payoff vectors in the game with patient players are typically a strict subset of the patient limit of the game with impatient players with equal discount rates.

[6] We adopt the following notational conventions throughout the paper. Given a finite set X, let ΔX denote the set of all probability measures over X. Given a probability measure $\mu \in \Delta X$, let $\text{supp}(\mu)$ denote its support. Given a set X in a real vector space, let $\text{co}(X)$ denote its convex hull. Given a real-valued function f, let $[f]_+$ denote $\max\{f(\cdot), 0\}$ and when the domain of f is convex, let $\text{vex} f$ denote the convexification of f (i.e., pointwise largest convex function that does not exceed f). Given a correspondence $V : X \rightrightarrows Y$, let $\text{gr}(V)$ denote the graph of V. Given a number $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x.

[7] For each $t \in \mathbb{N}$, let $\mathscr{H}^t$ be the finite algebra generated by the discrete algebra on A^t, and let $\mathscr{H}^\infty$ denote the product σ-algebra on A^∞. Then, the probability measure induced by (σ, μ) is a probability measure on the measurable space $(\Omega, \mathscr{H}^\infty \otimes 2^\Theta)$ that is uniquely defined by the Kolmogorov extension theorem.

[8] Aumann et al. (1968) and Hart (1985) refer to σ that satisfies (1.2) and (1.3) as a uniform equilibrium. See section 2 in Hart (1985) for other payoff-equivalent definitions of equilibrium.

[9] The limits exist given (1.2) and (1.3).

$$v_1(\sigma) := \lim_{t\to\infty} v_1^t(\sigma) \text{ and } v_2(\sigma) := \lim_{t\to\infty} v_2^t(\sigma). \tag{1.4}$$

A vector $s = (s_1, s_2) \in \mathbb{R}^2$ is an *equilibrium payoff vector* if an equilibrium σ exists such that $s = (v_1(\sigma), v_2(\sigma))$. For $j \in \{1, 2\}$, $s_j \in \mathbb{R}$ is an *equilibrium Pj-payoff* if an equilibrium σ exists for which $s_j = v_j(\sigma)$.

As in the models of Aumann et al. (1968) and Hart (1985), our model does not include a public randomization device. Such a coordination device is unnecessary because players can achieve the same equilibrium payoffs using jointly controlled lotteries.[10] Still, some modelers may prefer to include a public coordination device. We discuss in greater detail how our definitions and results can be adjusted to the environment in which players have access to a public randomization device in section 3.

2. Results

2.1. Informed player's equilibrium payoffs

Our main result is that any equilibrium payoff of player 1 can be obtained with a simple strategy profile in which the initial stages are used only for revealing information, and no further information is revealed after the initial stages. We also provide a condition that characterizes a payoff being an equilibrium $P1$-payoff in terms of the belief distribution that generates it.

Toward stating our main result, call an equilibrium σ an *AMS equilibrium* if some $\ell \in \mathbb{N}$ exists such that (i) players ignore player 2's behavior in the first ℓ stages, and (ii) σ_1 does not condition on θ for any on-path history after stage ℓ.[11] Observe, in particular, that the first condition prevents players from coordinating their actions using jointly controlled lotteries in the initial ℓ stages. Thus, in an AMS equilibrium, players use the initial finite (ℓ) number of stages solely for communication and not for coordination. We call an equilibrium payoff vector associated with an AMS equilibrium an *AMS-equilibrium payoff vector*. For $j \in \{1, 2\}$, $s_j \in \mathbb{R}$ is an *AMS-equilibrium Pj-payoff* if some AMS equilibrium σ exists such that $s_j = v_j(\sigma)$.

Let us now introduce a correspondence $F^*(\mu)$ that yields the set of payoffs that are feasible and individually rational given any belief $\mu \in \Delta\Theta$. This correspondence plays an important role as in the case of folk theorems for repeated games with complete information. Let $\bar{u}$ be a bound on the players' possible payoff magnitudes, define $\mathcal{R} := [-\bar{u}, \bar{u}] \subseteq \mathbb{R}$, and take $F : \Delta\Theta \rightrightarrows \mathcal{R}^2$ to be the correspondence that gives the set of feasible expected payoffs in the one-stage game from using a correlated state-independent strategy given any prior belief.[12] Let $\underline{u}_1 \in \mathbb{R}$ and $\underline{u}_2 : \Delta\Theta \to \mathbb{R}$ be the minmax values for players 1 and 2, respectively, in the one-stage game in which neither player observes the realization of state with a common prior belief.[13] We define $F^* : \Delta\Theta \rightrightarrows \mathcal{R}^2$ as

$$\mu \mapsto \left\{ (s_1, s_2) \in F(\mu) : \ s_1 \ge \underline{u}_1,\ s_2 \ge \operatorname{vex} \underline{u}_2(\mu) \right\},$$

and $F_1^* : \Delta\Theta \rightrightarrows \mathcal{R}$ as the projection of F^* to player 1's payoffs. Call a distribution over posterior beliefs, $p \in \Delta\Delta\Theta$, that averages to the prior μ_0 an *information policy* and let $\mathcal{I}(\mu_0) = \{p \in \Delta\Delta\Theta : \int_{\Delta\Theta} \mu \mathrm{d}p(\mu) = \mu_0\}$ denote the set of all information policies given a prior μ_0. We now formally state our main result below.

Proposition 1. *Given $s_1 \in \mathcal{R}$, the following are equivalent:*

(i) Payoff s_1 is an equilibrium $P1$-payoff.
(ii) Payoff s_1 is an AMS-equilibrium $P1$-payoff.
(iii) Some information policy $p \in \mathcal{I}(\mu_0)$ with finite support exists such that

$$p\left(\left\{\mu \in \Delta\Theta : s_1 \in F_1^*(\mu)\right\}\right) = 1. \tag{2.1}$$

(iv) Some information policy $p \in \mathcal{I}(\mu_0)$ exists such that (2.1) holds.

In the general payoff case, Aumann et al. (1968) provide examples of equilibrium payoff vectors that are not AMS-equilibrium payoff vectors. Hart (1985) subsequently provides a characterization of all equilibrium payoffs via strategy profiles that allow players to engage in more sophisticated communication and coordination (e.g., alternating between stages of communication and stages of coordination of actions via jointly controlled lotteries, possibly *ad infinitum*). In particular, Proposition 1implies that when player 1's preferences are state independent, the additional sophistication allowed under Hart (1985) is unnecessary from player 1's perspective.

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[10] See, for example, Forges (1994).

[11] Formally, an equilibrium σ is an AMS equilibrium if, for any $t \in \mathbb{N}$ with $t \ge \ell$, any pair of public histories $h, h' \in A^t$, and any pair of states $\theta, \theta' \in \Theta$, we have (i) if h and h' differ only in the first ℓ periods of player 2's play, then $\sigma_1(h, \theta) = \sigma_1(h', \theta)$ and $\sigma_2(h) = \sigma_2(h')$; and (ii) if h is reached with positive probability given σ, then $\sigma_1(h, \theta) = \sigma_1(h, \theta')$. Aumann et al. (1968) characterize the set of AMS equilibrium payoffs.

[12] That is, F maps μ to the set $\operatorname{co}(\{(u_1(a), \int_\Theta u_2(a, \cdot)\mathrm{d}\mu) : a \in A\})$.

[13] Let $\nu \otimes \nu'$ denote the product measure given any measures ν and ν'. Then,

$$\underline{u}_1 := \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} \int_A u_1(\cdot)\mathrm{d}(\alpha_1 \otimes \alpha_2),\ \underline{u}_2(\mu) := \min_{\alpha_1 \in \Delta A_1} \max_{\alpha_2 \in \Delta A_2} \int_{A\times\Theta} u_2(\cdot)\mathrm{d}(\alpha_1 \otimes \alpha_2 \otimes \mu).$$

Our proof begins with Hart's (1985) characterization of equilibrium payoff vectors via bimartingales, and we show player 1's payoff induced by such a bimartingale satisfies condition (2.1) for some information policy $p \in \mathcal{I}(\mu_0)$ giving us (iv). Because Θ is finite, Carathéodory's theorem implies p may be chosen to ensure that its support is finite, which gives us (iii). We then argue that (iii) implies player 1's payoff must be an AMS-equilibrium $P1$-payoff by recalling a characterization of AMS equilibrium as pointed out by Hart (1985). The proof is completed by noting that any AMS-equilibrium $P1$-payoff is an equilibrium $P1$-payoff by definition.

Formally, the starting point of our proof is a lemma that follows immediately from the Main Theorem in Hart (1985). To state it, define a *bimartingale* as a $\Delta\Theta \times \mathcal{R}$-valued martingale, $((\mu^t, s^t))_{t=1}^{\infty}$, on some filtered probability space such that, for all $t \in \mathbb{N}$, either $\mu^{t+1} = \mu^t$ almost surely or $s^{t+1} = s^t$ almost surely. We say a bimartingale has *initial value* (μ, s) if $(\mu^1, s^1) = (\mu, s)$ almost surely. Given a measurable subset Z of $\Delta\Theta \times \mathcal{R}$, we say a bimartingale has *terminal values in* Z if the almost-sure limit of the martingale is contained in Z almost surely.

Lemma 1. *If $s_1 \in \mathcal{R}$ is an equilibrium $P1$-payoff, then some bimartingale exists with initial value (μ_0, s_1) and terminal values in $\mathrm{gr}(F_1^*)$.*

The next lemma, which follows from a definition of AMS equilibrium as pointed out by Hart (1985),[14] gives a sufficient condition for a payoff $s_1 \in \mathcal{R}$ to be AMS-equilibrium $P1$-payoff.

Lemma 2. *Let $s_1 \in \mathcal{R}$. Suppose some $p \in \mathcal{I}(\mu_0)$ with finite support exists such that (2.1) holds. Then, s_1 is an AMS-equilibrium $P1$-payoff.*

Before proceeding with our next lemma, we record some useful facts proven in Lipnowski and Ravid (2020).

Fact 1. *Let $V : \Delta\Theta \rightrightarrows \mathbb{R}$ be a Kakutani correspondence and*

$$S_1 := \bigcup_{p \in \mathcal{I}(\mu_0)} \bigcap_{\mu \in \mathrm{supp}(p)} V(\mu).$$

Take any $s_1 \geq \max V(\mu_0)$ and $s_1' \leq \min V(\mu_0)$. Then,

(i) *S_1 is a nonempty compact interval.*
(ii) *$s_1 \in S_1$ if and only if $\mu_0 \in \overline{\mathrm{co}}(\{\mu \in \Delta\Theta : \max V(\mu) \geq s_1\})$.*
(iii) *$s_1' \in S_1$ if and only if $\mu_0 \in \overline{\mathrm{co}}(\{\mu \in \Delta\Theta : \min V(\mu) \leq s_1'\})$.*
(iv) *$\max S_1 = \bar{v}(\mu_0)$, where $\bar{v} : \Delta\Theta \to \mathbb{R}$ is the pointwise lowest quasiconcave function above* $\max V(\cdot)$.
(v) *$\min S_1 = \underline{v}(\mu_0)$, where $\underline{v} : \Delta\Theta \to \mathbb{R}$ is the pointwise highest quasiconvex function below* $\min V(\cdot)$.

Proof. Parts (i) and (ii) follow from analysis in Lipnowski and Ravid (2020), specifically, the proofs of Corollary 3 and Theorem 1, respectively. Part (iv) follows from Theorem 2 and Corollary 4 in Lipnowski and Ravid (2020). Parts (iii) and (v) follow from respectively applying parts (ii) and (iv) to the correspondence $-V$. □

We now state and prove Lemma 3 and Lemma 4 that link Lemma 1 to Lemma 2. The proof appeals to Aumann and Hart's (1986) characterization of the set of initial values of bimartingales using a separation concept—in particular, that the set of all initial values of bimartingale with terminal values contained in some closed set $Z \subsetneq \Delta\Theta \times \mathcal{R}$ is given by the set of points that cannot be separated from Z by any biconvex function[15] that is continuous on Z.[16]

Lemma 3. *Suppose $V : \Delta\Theta \rightrightarrows \mathbb{R}$ is a Kakutani correspondence.*[17] *If some bimartingale exists with initial value (μ_0, s_1) and terminal values in $\mathrm{gr}(V)$, then some $p \in \mathcal{I}(\mu_0)$ exists such that*

$$p\left(\left\{\mu \in \Delta\Theta : s_1 \in V(\mu)\right\}\right) = 1. \tag{2.2}$$

Proof. We prove the contrapositive statement following an argument in the proof of Proposition 4 in Lipnowski and Ravid (2020)—which we include here for the sake of self-contained presentation. Let S_1 denote the set of informed player's payoffs such that (2.2) holds for some $p \in \mathcal{I}(\mu_0)$, and suppose $s_1 \in \mathcal{R} \setminus S_1$. Because V is a Kakutani correspondence, Fact 1 says S_1 is a compact interval $[\underline{s}_1, \bar{s}_1]$, and so either $s_1 > \bar{s}_1$ or $s_1 < \underline{s}_1$. Focusing on the first case (the argument for the second case being analogous), fix a payoff $s_1' \in (\bar{s}_1, s_1)$. Fact 1 implies μ_0 is not in the set $B := \overline{\mathrm{co}}(\{\mu \in \Delta\Theta : \max V(\mu) \geq s_1'\})$. Hence, the Hahn-Banach theorem delivers

[14] See the second paragraph of section 6 in Hart (1985).

[15] Given convex spaces X and Y, a subset of the product space $X \times Y$ is biconvex if all its x- and y-sections are convex. A real-valued function on a biconvex subset of $X \times Y$ is biconvex (resp. bi-affine) if it is convex (resp. affine) in each variable x and y separately.

[16] One can view Aumann and Hart's (1986) characterization as a biconvex analogue of standard convex separation results. In particular, that the set of initial values of martingales with terminal values contained in Z is given by $\mathrm{co}(Z)$, which is also the set of points that cannot be separated from Z by any convex function that is continuous on Z.

[17] A Kakutani correspondence is a nonempty-, convex-, compact-valued and upper hemicontinuous correspondence.

an affine continuous $\varphi : \Delta\Theta \to \mathbb{R}$ that separates μ_0 from B; i.e., $\varphi(\mu_0) > \max \varphi(B)$. Now define a function $b : \Delta\Theta \times \mathcal{R} \to \mathbb{R}_+$ via $b(\mu, x) := [\varphi(\mu) - \max \varphi(B)]_+ [x - x']_+$. Observe that b is a continuous and biconvex function that separates (μ_0, s_1) from $\mathrm{gr}(V)$; i.e., $b(\mu_0, s_1) > 0$ by the choice of s_1', and $b|_{\mathrm{gr}(V)} = 0$ because either $\mu \notin B$ in which case $s_1 < s_1'$ or because $\mu \in B$ so that $\varphi(\mu) \le \max \varphi(B)$. Therefore, by Theorem 4.7 in Aumann and Hart (1986), there does not exist a bimartingale with initial value (μ_0, s_1) and terminal values in $\mathrm{gr}(V)$. □

The next lemma shows we may substitute F_1^* as V in the lemma above.

Lemma 4. *The correspondence F_1^* is a Kakutani correspondence.*

Proof. Let $\hat{F} : \Delta\Theta \rightrightarrows \mathcal{R}^2$ denote the correspondence $\mu \mapsto \{(u_1(a), \int_\Theta u_2(a, \cdot)\mathrm{d}\mu) : a \in A\}$. Observe that $\hat{F}$ is compact-valued (because a finite set is compact), and because its graph is closed (being a union of the graphs of finitely many continuous functions), $\hat{F}$ is upper hemicontinuous. As a convex hull of the real-valued correspondence $\hat{F}$, the correspondence F is convex- and compact-valued and upper hemicontinuous. Define $\tilde{F} : \Delta\Theta \rightrightarrows \mathcal{R}^2$ as the individual rationality correspondence $\mu \mapsto \{(s_1, s_2) \in \mathcal{R}^2 : s_1 \ge \underline{u}_1,\ s_2 \ge \mathrm{vex}\, \underline{u}_2(\mu)\}$. Observe that $\tilde{F}$ is convex- and compact-valued (taking values in the bounded set $\mathcal{R}^2$). Moreover, $\tilde{F}$ is upper hemicontinuous because $\mathrm{vex}\, \underline{u}_2$ is lower semicontinuous. Because $\hat{F}$ is convex-valued with a closed graph, whereas $\tilde{F}$ is convex-valued with a compact graph, their intersection is convex-valued with a compact graph. Hence, F^* is convex-, compact-valued, and upper hemicontinuous. For any $\mu \in \Delta\Theta$, the set $F^*(\mu)$ contains the payoff vector associated with player 1 playing the minmax mixed strategy for u_1 and player 2 playing a minmax mixed strategy for $\int_\Theta u_2(\cdot, \theta)\mathrm{d}\mu(\theta)$, and so F^* is nonempty-valued. Therefore, F^* is a Kakutani correspondence. Because F_1^* is a projection of F^* to the first coordinate, which is a continuous transformation, F_1^* is a Kakutani correspondence. □

Note that the proof of Lemma 4 shows that establishing the existence of an equilibrium (i.e., F_1^* is non-empty valued) with state-independent informed-player preferences is trivial, unlike in the case with general payoffs.[18] We are now ready to prove our main result, Proposition 1.

Proof of Proposition 1. Lemma 2 says (iii) implies (ii); and (ii) directly implies (i). Now suppose (iv) holds; let us see it implies (iii). Given (iv), the prior is in the closed convex hull of $\{F_1^* \ni s_1\} := \{\mu \in \Delta\Theta : s_1 \in F_1^*(\mu)\}$. Note, also, that $\{F_1^* \ni s_1\}$ is compact, because F_1^* has a compact graph (by Lemma 4). Therefore, the closed convex hull of $\{F_1^* \ni s_1\}$ equals the convex hull of $\{F_1^* \ni s_1\}$—this is a well-known consequence of Carathéodory's theorem (see, for example, Theorem 17.2 in Rockafellar, 1997). Part (iii) then follows from the definition of the convex hull. Finally, to see (i) implies (iv), let $s_1 \in \mathcal{R}$ be an equilibrium $P1$-payoff. Lemma 1 delivers a bimartingale with initial value (μ_0, s_1) and terminal values in $\mathrm{gr}(F_1^*)$. Then, by Lemma 3 and 4, some $p \in \mathcal{I}(\mu_0)$ exists that satisfies (2.1)—that is, (iv) holds too. □

To conclude the subsection, we note a consequence of the sufficiency of AMS equilibria: the set of equilibrium $P1$-payoffs can be computed directly by quasiconvexifying the correspondence F_1^*.

Corollary 1. *The set of equilibrium $P1$-payoffs is $[\underline{f}(\mu_0), \overline{f}(\mu_0)]$, where $\underline{f}$ is the pointwise largest quasiconvex function below $\min F_1^*(\cdot)$, and $\overline{f}$ is the pointwise largest quasiconcave function above $\max F_1^*(\cdot)$.*

Proof. Proposition 1 establishes that s_1 is an equilibrium $P1$-payoff if and only if some $p \in \mathcal{I}(\mu_0)$ exists that satisfies (2.1). The result then follows from Lemma 4 and Fact 1. □

The following example demonstrates how our characterization of equilibrium $P1$-payoffs and Corollary 1 constitute a substantial gain in tractability in computing the set of informed player's equilibrium payoffs.

Example 1. Consider a seller of two essential oils, eucalyptus and rosehip, interacting with a repeat buyer concerned about longevity. Although aromatherapy using the appropriate oil will prolong the life of the buyer,[19] he does not know which oil is effective. The seller, who knows which of the two oils is beneficial for the buyer, will recommend one of the two oils in each period. The seller does not incur any cost in making recommendations (nor does the recommendation directly affect the buyer's payoffs). Suppose the seller's profit from selling either oil is positive but selling rosehip is more profitable than eucalyptus oil. To capture such a situation, label the seller as player 1 and the buyer as player 2. Let $\theta = 1$ (resp. $= 0$) denote the state in which rosehip (resp. eucalyptus) is effective for the buyer. Assume that each oil is initially believed to be equally likely to increase longevity. Let 1, 2, 0 denote the buyer respectively purchasing eucalyptus oil, rosehip oil, or not buying either, and e, r denote the seller respectively recommending eucalyptus oil or rosehip oil. The seller's payoff is simply $a_2 \in \{0, 1, 2\}$. The buyer, on the other hand, would like to purchase the oil

[18] Simon et al. (1995) provides a proof of the existence of an equilibrium in the general payoff case.

[19] This paper does not provide medical advice. The reader should consult a naturopath (or a doctor).

Players' payoffs: $(u_1(\cdot, a_2),\ u_2(\cdot, a_2, \theta))$

	$a_2 = 1$	0	2
$\theta = 0$	1, 1	0, 0	2, −3
1	1, −3	0, 0	2, 1

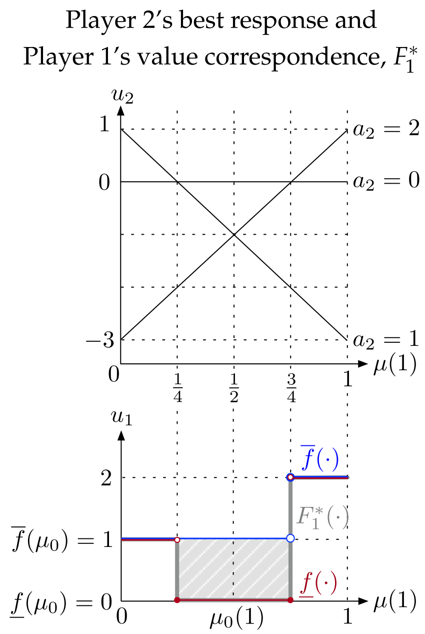

Fig. 2.1. Aromatherapy. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

that increases longevity with sufficiently high probability and otherwise not purchase anything at all; in particular, the buyer buys rosehip (resp. eucalyptus) oil only if he believes with probability above $\frac{3}{4}$ that $\theta = 1$ (resp. $\theta = 0$) meaning that the oil that in fact increases longevity is rosehip (resp. eucalyptus). Fig. 2.1 shows the payoffs associated with each action under each state (note seller's action $a_1 \in \{e, r\}$ does not affect payoffs), the buyer's best response as a function of his belief that the effective oil is rosehip, $\mu(1)$, and the seller's value correspondence, F_1^*, and functions $\overline{f}$ (in blue) and $\underline{f}$ (in red) as defined in Corollary 1.

Corollary 1 immediately gives that the set of equilibrium payoffs for the buyer is the interval $[0, 1]$.[20] □

2.2. *Equilibrium payoff vectors and the uninformed player's equilibrium payoffs*

We emphasize that Proposition 1 applies to the informed player's equilibrium payoffs and not the equilibrium payoff vectors or the uninformed player's equilibrium payoffs. To demonstrate, we adapt examples with state-independent sender preferences from the cheap-talk literature.[21]

The first example, adapted from Aumann and Hart (2003), demonstrates some equilibrium payoff vector exists that is not an AMS-equilibrium payoff vector. Nevertheless, the set of equilibrium Pj-payoffs in this example coincides with the set of AMS-equilibrium Pj-payoffs for each player $j \in \{1, 2\}$. The second example is adapted from Lipnowski and Ravid (2020). We show this example admits an equilibrium $P2$-payoff that is not attainable in any AMS equilibrium. The third and the last examples show that equilibrium payoff vectors and equilibrium $P2$-payoffs exist that cannot be attained if the number of communication stages is bounded. Taken together, the examples demonstrate that, whereas assuming player 1's preferences are state independent simplifies the characterization of equilibrium $P1$-payoffs, the same simplification does not apply to the set of attainable equilibrium payoff vectors or the set of equilibrium $P2$-payoffs.

Example 2 *(Example 2.6 in Aumann and Hart, 2003).* There are two equally likely states, $\Theta := \{0, 1\}$, and player 2 has five actions, $A_2 := \{LL, L, C, R, RR\}$. Fig. 2.2 shows the payoffs associated with each action under each state, player 2's best response as a function of his belief that the state is 1, $\mu(1)$, and player 1's value correspondence, F_1^*. □

[20] More generally, given binary state $\Theta := \{0, 1\}$, Corollary 1 implies the set of equilibrium $P1$-payoffs is the interval $[\underline{f}(\mu_0), \overline{f}(\mu_0)]$, where

$$\underline{f}(\mu_0) = \max\left\{ \min_{\mu:\ \mu(1)\in[0,\mu_0(1)]} \min F_1^*(\mu),\ \min_{\mu:\ \mu(1)\in[\mu_0(1),1]} \min F_1^*(\mu) \right\},\ \overline{f}(\mu_0) = \min\left\{ \max_{\mu:\ \mu(1)\in[0,\mu_0(1)]} \max F_1^*(\mu),\ \max_{\mu:\ \mu(1)\in[\mu_0(1),1]} \max F_1^*(\mu) \right\}.$$

Thus, the ability to restrict attention to AMS equilibria constitutes a substantial tractability gain in computing the range of equilibrium $P1$-payoffs.

[21] Unlike in cheap-talk games, our environment has no explicit communication technology. We therefore adapt the original examples by allowing the informed player to have at least two actions that lead to the same stage payoffs.

Players' payoffs: $(u_1(\cdot, a_2),\, u_2(\cdot, a_2, \theta))$

	$a_2 = LL$	L	C	R	RR
$\theta = 0$	1, 10	3, 8	0, 5	3, 0	1, −8
1	1, −8	3, 0	0, 5	3, 8	1, 10

Player 2's best response and
Player 1's value correspondence, F_1^*

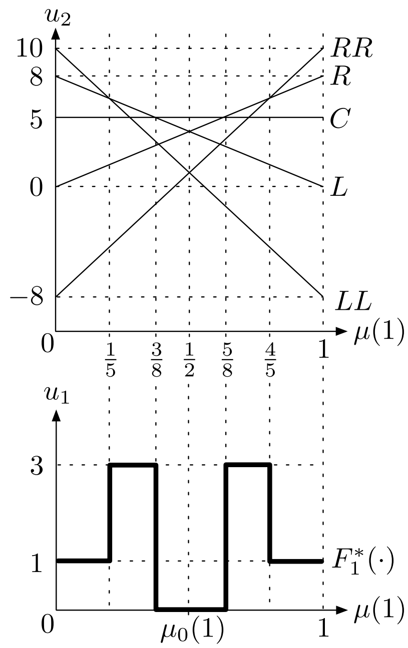

Fig. 2.2. Example 2.6 in Aumann and Hart (2003).

By Proposition 1, any $s_1 \in [0, 3]$ is an equilibrium $P1$-payoff if it satisfies (2.1) for some information policy. Since F_1^* is symmetric around the prior, it follows that any $s_1 \in [0, 3]$ can be obtained by a binary-support information policy that satisfies (2.1). Thus, any $s_1 \in [0, 3]$ is an equilibrium $P1$-payoff.[22]

Example 3. We now claim that $(2, 8)$ is an equilibrium payoff vector of this game that is not an AMS-equilibrium payoff vector. As explained in Aumann and Hart (2003), $(2, 8)$ can be attained in an equilibrium in which players first perform a jointly controlled lottery with equal probabilities,[23] and depending on the outcome of the jointly controlled lottery, player 1 either fully reveals the state yielding a payoff of $(1, 10)$,[24] or partially reveals the state such that player 2's posterior belief, $\mu(1)$, is either $\frac{1}{4}$ or $\frac{3}{4}$ yielding a payoff of $(3, 6)$.[25] To see why $(2, 8)$ cannot be an AMS-equilibrium payoff vector, note first that, because players do not ignore player 2's action in responding to the lottery, jointly controlled lotteries can only occur after the initial communication stage in any AMS-equilibrium. To induce a payoff of 2 for player 1 in an AMS equilibrium, the posterior belief for player 2 must therefore always be one of $\frac{1}{5}$, $\frac{3}{8}$, $\frac{5}{8}$, or $\frac{4}{5}$. However, with such beliefs, player 2's expected payoffs are strictly below 8. Hence, it follows that $(2, 8)$ cannot be an AMS-equilibrium payoff vector. □

In the previous example, observe that the set of equilibrium $P2$-payoffs and the set of AMS-equilibrium $P2$-payoffs coincide.[26] The following example shows that this observation does not hold generally.

Example 4 *(Appendix C.3 in Lipnowski and Ravid, 2020)*. There are two possible states, $\Theta := \{0, 1\}$, and the prior belief is that the state is 1 with probability $\frac{1}{8}$. Player 2 has four actions, $A_2 := \{\ell, b, t, r\}$. Fig. 2.3 shows the payoffs associated with each action under each state, player 2's best response as a function of his belief that the state is 1, $\mu(1)$, and player 1's value correspondence, F_1^*. By Proposition 1 (because $F_1^*(\mu) = \{1\}$ for any $\mu(1) \le \frac{1}{8}$) or by Corollary 1 (because $[\underline{f}(\mu_0), \overline{f}(\mu_0)] = \{1\}$), player 1's payoff must be 1 in any AMS equilibrium (and, hence, this is also the unique equilibrium $P1$-payoff). Moreover, in any AMS equilibrium, player 2's

22 The result is also immediate from Corollary 1 because $[\underline{f}(\mu_0), \overline{f}(\mu_0)] = [0, 3]$ in the example.

23 For example, player 1 chooses first-stage action uniformly and player 2 chooses first-stage action uniformly among $\{LL, RR\}$. Then, player 1 fully reveals if and only if $a_1^1 = 1$ or $a_2^1 = LL$, and partially reveals (in the manner specified below) if $a_1^1 = 0$ and $a_2^1 = RR$. Notice how the jointly controlled lottery is playing the role of a public randomization device.

24 For example, player 1 chooses $a_1^2 = \theta$ if and only if the state is $\theta \in \Theta$.

25 For example, player 1 chooses $a_1^2 = 1$ with probability $\frac{1}{4}$ if the state is 1 and with probability $\frac{3}{4}$ if the state is 0.

26 To see this, observe that, in any equilibrium, player 2's payoff must lie in $[5, 10]$. As already mentioned, any $s_1 \in [0, 3]$ can be achieved by inducing a symmetric posterior belief around $\frac{1}{2}$. Finally, observe that such a distribution over posterior beliefs can induce any expected payoff for player 2 in $[5, 10]$.

Players' payoffs: $(u_1(\cdot, a_2),\, u_2(\cdot, a_2, \theta))$

	$a_2 = \ell$	b	t	r
$\theta = 0$	$1, 0$	$0, -\frac{1}{2}$	$2, -\frac{1}{4}$	$2, -1$
1	$1, -1$	$0, -\frac{1}{4}$	$2, -\frac{1}{4}$	$2, 0$

Player 2's best response and
Player 1's value correspondence, F_1^*

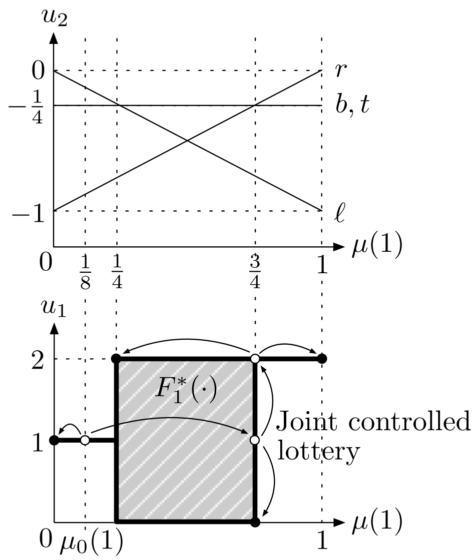

Fig. 2.3. Example in Appendix C.3 in Lipnowski and Ravid (2020).

maximum payoff is $-\frac{1}{24}$, corresponding to a distribution over beliefs $\mu(1) = 0$ with probability $\frac{5}{6}$ and $\mu(1) = \frac{3}{4}$ with probability $\frac{1}{6}$.[27] However, as explained in Lipnowski and Ravid (2020), players may perform a jointly controlled lottery (with equal probabilities) following realization of posterior belief $\frac{3}{4}$ (vertical arrows in the figure), and player 1 could further communicate (upper horizontal arrows in the figure) so that player 2's payoff will be supported by the solid dots as shown in the figure, which must yield a strictly higher payoff than $-\frac{1}{24}$. Such splits (called diconvexifications) are allowed under Hart's (1985) characterization, and it follows that the resulting payoff for player 2 is an equilibrium $P2$-payoff. Thus, some equilibrium $P2$-payoff exists that is not an AMS-equilibrium $P2$-payoff. □

In the case when the *un*informed player's preference is state independent, Shalev (1994) shows that every equilibrium payoff vector is attainable in some AMS equilibrium in which the informed player fully reveals the state to the uninformed player. In contrast, when it is the informed player's preference that is state independent, not all equilibrium $P1$-payoffs can be attained in a fully revealing AMS equilibrium. For instance, Example 2 shows that attainable payoffs for the informed player (e.g., $s_1 = 5$) exist that are unattainable in a fully revealing AMS equilibrium. Example 4 is even more extreme in that it does not admit any fully revealing AMS equilibrium. Consequently, none of the informed player's equilibrium payoffs can be attained in a fully revealing AMS equilibrium.

The payoffs considered in the previous examples, while not achievable as an AMS-equilibrium payoff vector or equilibrium $P2$-payoff, are nevertheless achievable through finite alternations of stages of communication followed by stages of coordination. The next example demonstrates that, even when the informed player's payoffs are known, some equilibrium $P2$-payoff exists that cannot be obtained if the alternations are bounded. The example is inspired by Forges' (1984) "four frogs" game that generates the geometric structure from Example 2.5 in Aumann and Hart (1986). An implication of the example is that payoff vectors from equilibria in which players stop communicating information after a finite number of stages do not characterize the set of equilibrium payoff vectors (even) when the informed player's payoffs are known.

Example 5 *("Four Frogs" with known informed player's payoffs).* There are two equally likely states, $\Theta := \{0, 1\}$, and player 2 has six actions, $A_2 := \{\ell\ell, \ell, b, t, r, rr\}$. Fig. 2.4 shows the payoffs associated with each action under each state, player 2's best response as a function of his belief that the state is 1, $\mu(1)$, and player 1's value correspondence, F_1^*.

By Proposition 1 (or Corollary 1), it is immediate from the figure that the interval $[\frac{1}{3}, \frac{2}{3}]$ is the set of equilibrium $P1$-payoffs. Observe further that in any AMS equilibrium in which player 1 obtains a payoff of $s_1 \in (\frac{1}{3}, \frac{2}{3})$, player 2's payoff must be zero since any information policy that satisfies condition (2.1) must have support in $[\frac{1}{3}, \frac{2}{3}]$. However, an equilibrium exists with equilibrium

[27] For example, player 1 chooses $a_1^1 = 1$ with probability 1 if the state is 1 and with probability $\frac{1}{21}$ if the state is 0.

Players' payoffs: $(u_1(\cdot, a_2),\, u_2(\cdot, a_2, \theta))$

	$a_2 = \ell\ell$	ℓ	b	t	r	rr
$\theta = 0$	$\frac{2}{3}, 4$	$0, 2$	$\frac{2}{3}, 0$	$\frac{1}{3}, 0$	$1, -4$	$\frac{1}{3}, -8$
1	$\frac{2}{3}, -8$	$0, -4$	$\frac{2}{3}, 0$	$\frac{1}{3}, 0$	$1, 2$	$\frac{1}{3}, 4$

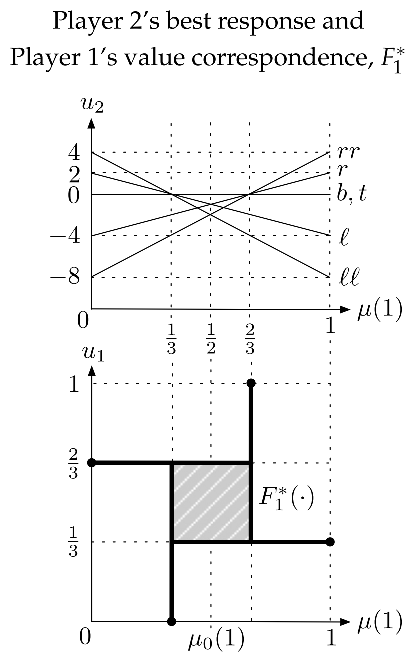

Fig. 2.4. "Four Frogs" with known informed player's payoffs.

$P1$-payoff of $\frac{2}{3}$ in which player 2's payoff is $1 > 0$.[28] Moreover, one can show $s_2 = 1$ is the maximum payoff that player 2 can attain in any AMS equilibrium.

We will now argue that player 2 can in fact obtain a payoff of 2 in an equilibrium and that this payoff can be attained only with infinite stages of communication. Define $L \subseteq \Delta\Theta \times [0,1]$ to be the set consisting of the "lily pads" on which the frogs can "land"; i.e., a collection of points in $\Delta\Theta \times [0,1]$ corresponding to $\{(0, \frac{2}{3}), (\frac{1}{3}, 0), (\frac{2}{3}, 1), (1, \frac{1}{3})\}$ in the second figure. We claim that an equilibrium that yields player 1 a payoff of s_1 gives the highest payoff for player 2 if and only if a bimartingale exists with initial value (μ_0, s_1) with terminal values in L. As is well known (Forges, 1984; Aumann and Hart, 1986), no bimartingale with finite stages of communication can be supported on L. It follows that no finite number of stages of communication is sufficient to yield the player 2 preferred equilibrium payoff.

Toward proving the claim, define a bi-affine function $\bar{u}_2 : \mathrm{gr}(F_1^*) \to \mathbb{R}$ that is continuous on $\mathrm{gr}(F_1^*)$ as $\bar{u}_2(\mu, s_1) := 2 - 6[2\mu(1) - 1](2s_1 - 1)$. Let $u_2^* : \Delta\Theta \to \mathbb{R}$ be such that $u_2^*(\mu)$ denotes player 2's payoff from best responding to belief $\mu \in \Delta\Theta$ so that $u_2^*(\mu) = [|12\mu(1) - 6| - 2]_+$. Notice that $\bar{u}_2(\mu, s_1) = u_2^*(\mu)$ for any $(\mu, s_1) \in L$ and $\bar{u}_2(\mu, s_1) > u_2^*(\mu)$ for any $(\mu, s_1) \in \mathrm{gr}(F_1^*) \backslash L$.[29] Consider any bimartingale $((\mu^t, s^t))_{t=1}^{\infty}$ with initial value (μ_0, s_1). Then,

$$\mathbb{E}\left[u_2^*\left(\mu^{\infty}\right)\right] \leq \mathbb{E}\left[\bar{u}_2\left(\mu^{\infty}, s_1^{\infty}\right)\right] = \lim_{t\to\infty} \mathbb{E}\left[\bar{u}_2\left(\mu^t, s^t\right)\right] = \bar{u}_2\left(\mu_0, s_1\right) = 2,$$

where $\mu^{\infty} := \lim_{t\to\infty} \mu^t$ and $s_1^{\infty} := \lim_{t\to\infty} s_1^t$. The first and second equalities in the above chain follow from $\bar{u}_2$ being continuous and bi-affine, respectively. Observe further that the inequality holds with equality if and only if $(\mu^{\infty}, s_1^{\infty}) \in L$ almost surely. □

3. Discussion

In this section, we briefly discuss the implications of introducing a public randomization device to our model, the length of the communication phase necessary in an AMS equilibrium, and an application of our main result to patient limits of games with impatient players.

Public randomization device As explained above, our game does not include a public randomization device. However, some modelers may wish to include such devices in their games. We now explain what happens if we modify our game to include a public randomization device. Clearly, for every equilibrium in the original game, there is a corresponding equilibrium of the modified game. Therefore, we can define an AMS equilibrium in the modified game to be any equilibrium that is attainable as an AMS equilibrium in the original game. Formally, this definition reduces to requiring the equilibrium to satisfy conditions (i) and (ii) from the original

[28] Induced by an information policy with support $\{0, \frac{2}{3}\}$.

[29] To see the latter, observe first that every (μ, s_1) such that $(\mu(1), s_1) \in [\frac{1}{3}, \frac{2}{3}]^2$ has $\bar{u}_2(\mu, s_1) \geq 2 - 6 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{4}{3} > 0 = u_2^*(\mu)$. Consequently, every other $(\mu, s_1) \in \mathrm{gr}(F_1^*) \backslash L$ is in the interior of a line segment on which $\bar{u}_2$ is affine and above but not identically equal to the convex function $(\tilde{\mu}, \tilde{s}_1) \mapsto u_2^*(\tilde{\mu})$—and therefore has $\bar{u}_2(\mu, s_1) > u_2^*(\mu)$.

definition, as well as a requirement that no player ever conditions their actions on the public randomization device. Using this definition, Proposition 1 extends to the modified game; i.e., in the modified game, every equilibrium $P1$-payoff is an AMS-equilibrium $P1$-payoff. The reason is that every equilibrium payoff vector attainable with the use of such a public randomization device is also attainable without it; instead of relying on public randomization, players can use jointly controlled lotteries. In other words, adding a public randomization device does not expand the set of equilibrium payoffs.

Length of communication phase Recall that the definition of an AMS equilibrium does not specify a bound on the number of stages, ℓ, that is used for communication; i.e., ℓ can be any finite number of periods. The reason is purely combinatorial. The required length of the communication phase depends on the number of actions the informed player can use as signals vis-à-vis the support of the information policy associated with an AMS equilibrium. In particular, if a target information policy $p \in \mathcal{I}(\mu_0)$ needs to be supported on n beliefs, then ℓ has to be at least $\lceil \log_{|A_1|}(n) \rceil$ (and any such ℓ can implement an information policy with n elements in its support). It therefore follows that the communication phase in the AMS equilibrium can be one period long (i.e., $\ell = 1$) if $|A_1| \geq |\Theta|$.

Patient limits of games with impatient players Our analysis above focuses on patient players, and we adopt Aumann and Maschler's (1966) approach of players caring only about their long-run average payoffs. An alternative approach is to consider the patient limit of environments with impatient players; that is, players who care about their expected discounted payoffs but discount future payoffs by very little. In the appendix, we provide conditions under which these patient limits include the set of payoffs identified in Proposition 1.

CRediT authorship contribution statement

Takuma Habu: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Elliot Lipnowski:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Doron Ravid:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

None

Data availability

No data was used for the research described in the article.

Appendix A. Patient limit of games with impatient players

We now modify the game from the main body of the paper by assuming that players discount future payoffs while maintaining all other features. Specifically, for any $\delta \in (0,1)$, we say that a strategy profile σ is a *δ-equilibrium* if σ is a sequential equilibrium of the infinitely repeated game in which players 1 and 2's expected payoffs are evaluated, respectively, as

$$\tilde{v}_1(\sigma,\delta) := \mathbb{E}_{\sigma,\mu_0}\left[(1-\delta)\sum_{t=1}^{\infty}\delta^t u_1\left(a^t\right)\right],\ \tilde{v}_2(\sigma,\delta) := \mathbb{E}_{\sigma,\mu_0}\left[(1-\delta)\sum_{t=1}^{\infty}\delta^t u_2\left(a^t,\theta\right)\right].$$

A vector $s = (s_1, s_2) \in \mathbb{R}^2$ is a *δ-equilibrium payoff vector* if a δ-equilibrium σ exists such that $s = (\tilde{v}_1(\sigma,\delta), \tilde{v}_2(\sigma,\delta))$. For $j \in \{1,2\}$, $s_j \in \mathbb{R}$ is a *δ-equilibrium Pj-payoff* if a δ-equilibrium σ exists for which $s_j = \tilde{v}_j(\sigma,\delta)$.

The following claim provides conditions under which any equilibrium $P1$-payoff (in the game with patient players) can be approximated by a patient limit (i.e., $\delta \to 1$) of δ-equilibrium $P1$-payoffs (in the game with impatient players).

Claim A.1. Suppose $\underline{u}_2(\cdot)$ is convex and $F^*(\mu)$ has nonempty interior for all $\mu \in \Delta\Theta$. Then:

(i) The set of equilibrium $P1$-payoffs (in the game with patient players) for prior μ_0 is a nondegenerate interval $[\underline{s}_1, \overline{s}_1]$.
(ii) Any $s_1 \in (\underline{s}_1, \overline{s}_1)$ admits $\underline{\delta} \in [0,1)$ such that s_1 is a δ-equilibrium $P1$-payoff for all $\delta \in [\underline{\delta}, 1)$.

Toward proving the claim above, define $F^{**} : \Delta\Theta \rightrightarrows \mathcal{R}^2$ as

$$\mu \mapsto \left\{\left(s_1, s_2\right) \in F(\mu) : s_1 \geq \underline{u}_1,\ s_2 \geq \underline{u}_2(\mu)\right\}.$$

We first show that any payoff vector in the interior of F^{**} can be attained as a non-revealing δ-equilibrium of a game with prior μ for sufficiently high δ.

Lemma A.1. *Suppose $\mu \in \Delta\Theta$ and $s^\mu \in \mathcal{R}^2$ is an interior point in $F^{**}(\mu)$. Then, some $\epsilon > 0$ and $\underline{\delta} \in (0,1)$ exist such that every $s \in \mathcal{R}^2$ with $\|s - s^\mu\|_\infty \le \epsilon$ and every $\delta \in [\underline{\delta}, 1)$ admit some non-revealing δ-equilibrium $\sigma(\mu, s, \delta)$ of game with prior μ and discount factor δ with δ-equilibrium payoff vector s.*

Proof. In what follows, say (given fixed μ) that $\underline{\delta} \in (0,1)$ is *patient enough* for $s \in \mathcal{R}^2$ if every $\delta \in [\underline{\delta}, 1)$ admit some non-revealing δ-equilibrium $\sigma(\mu, s, \delta)$ of game with prior μ and discount factor δ with payoff vector s. We aim to show some $\epsilon > 0$ and $\underline{\delta} \in (0,1)$ exist such that $\underline{\delta}$ is patient enough for every $s \in \mathcal{R}^2$ with $\|s - s^\mu\|_\infty \le \epsilon$.

Observe first that there is a one-to-one correspondence between non-revealing equilibria of the incomplete information game with prior μ and equilibria of the complete information game in which player 2's payoffs are given by $\mathbb{E}_\mu[u_2(\cdot, \theta)]$. We now explain how to prove the result by appealing to the mixed-minmax perfect monitoring theorem for the complete information game. Let $\operatorname{int} F^{**}(\mu)$ denote the interior of the set $F^{**}(\mu) \subset \mathbb{R}^2$. For any $s \in \operatorname{int} F^{**}(\mu)$, Proposition 3.8.1 in Mailath and Samuleson (2006) delivers $\underline{\delta}(s) \in (0,1)$ such that every $\delta \in (\underline{\delta}(s), 1)$ has a subgame perfect equilibrium (without a public randomization device) of the repeated game with discount factor δ giving payoffs s—or equivalently, that $\underline{\delta}(s)$ is patient enough for s. It remains to argue that we may pick the same $\underline{\delta}$ for every $s \in \mathcal{R}^2$ with $\|s - s^\mu\|_\infty \le \epsilon$ for some $\epsilon > 0$. To make this argument, we rely on the proof of Proposition 3.8.1 in Mailath and Samuleson (2006), which identifies a $\underline{\delta}(s)$ that is in fact patient enough for all $\tilde{s} \in \mathcal{R}^2$ such that $\tilde{s} \ge s$.[30] To see how to use Mailath and Samuleson's (2006) proof to make our argument, note that because $s^\mu \in \operatorname{int} F^{**}(\mu)$, we can always find an $\epsilon > 0$ such that $s^* = (s_1^\mu - \epsilon, s_2^\mu - \epsilon) \in \operatorname{int} F^{**}(\mu)$. Therefore, we can apply the proof of Proposition 3.8.1 in Mailath and Samuleson (2006) to $s^* \in \operatorname{int} F^{**}(\mu)$ to obtain a $\underline{\delta}(s^*)$ that is patient enough for s^*. Because $s \ge s^*$ for all $s \in \mathcal{R}^2$ such that $\|s - s^\mu\|_\infty \le \epsilon$, we conclude that the $\underline{\delta}(s^*)$ has the desired properties for all such vectors s. □

Proof of Claim A.1. (i) That the set of equilibrium $P1$-payoff is a compact interval follows from Corollary 1. To see this interval is nondegenerate, note that it contains $F_1^*(\mu_0)$, which has nonempty interior because $F^*(\mu_0)$ does by hypothesis.

(ii) Take any $s_1 \in (\underline{s}_1, \overline{s}_1)$. Below, we argue some finite-support $p \in \mathcal{I}(\mu_0)$ exists such that every μ in the support of p, denoted D, has $s_1 \in \operatorname{int} F_1^*(\mu)$. Before doing so, let us explain how the existence of such a p enables the construction of a δ-equilibrium yielding a $P1$-payoff of s_1 for all sufficiently high discount factors δ. Note first that, for each $\mu \in D$, we have $s_1 \in \operatorname{int} F_1^*(\mu)$ and $F^*(\mu)$ is convex. Therefore, each $\mu \in D$ admits some s_2^μ such that $s^\mu = (s_1, s_2^\mu) \in \operatorname{int} F^*(\mu)$. But then, Lemma A.1 delivers some $\epsilon^\mu > 0$ and $\underline{\delta}^\mu \in (0,1)$ such that every payoff vector s with $\|s - s^\mu\|_\infty \le \epsilon^\mu$ admits a non-revealing equilibrium of the game with prior μ and any discount factor $\delta \ge \underline{\delta}^\mu$ that yields payoff vector s. For each $s_1' \in [s_1 - \epsilon^\mu, s_1 + \epsilon^\mu]$ and $\delta \in [\underline{\delta}^\mu, 1)$, let $\sigma(s_1', \mu, \delta)$ denote said non-revealing δ-equilibrium generating payoff vector (s_1', s_2^μ). Now, let $\ell = \lceil \log_{|A_1|} |D| \rceil$, and let $\underline{\delta} \in (0,1)$ be such that $\underline{\delta} \ge \underline{\delta}^\mu$ for every $\mu \in D$ and

$$\left| \underline{\delta}^{-\ell} \left[s_1 - \left(1 - \underline{\delta}^\ell\right) u_1(a) \right] - s_1 \right| < \epsilon^\mu \;\forall a \in A,\; \forall \mu \in D.$$

Such a $\underline{\delta}$ exists because D and A are both finite and the left-hand side of the above inequality converges to zero as $\underline{\delta} \to 1$. Because the ϵ^μ-neighborhood of s_1 is convex, it follows that $\tilde{s}_1((a^t)_t) = \delta^{-\ell}[s_1 - (1-\delta)\sum_{t=1}^{\ell} \delta^{t-1} u_1(a^t)]$ lives in this neighborhood for any sequence of action profiles $(a^t)_{t=1}^{\ell} \in A^\ell$. We can therefore construct our δ-equilibrium (for $\delta \ge \underline{\delta}$) as follows. In the first ℓ periods, player 1 plays in a way (while ignoring player 2's actions) that generates belief distribution p for player 2; without loss, have her do so in such a way that all action sequences in A_1^ℓ have strictly positive probability. In each of the first ℓ periods, have player 2 choose any myopic best response to any consistent beliefs (which exists because A_2 is finite). Now, let $\mu((a_1^t)_t) \in D$ denote player 2's belief (derived from Bayesian updating) following $(a_1^t)_t \in A_1^\ell$. Then, following any sequence of action profiles $(a^t)_{t=1}^{\ell} \in A^\ell$, the continuation strategy profile is given by $\sigma(\tilde{s}_1((a^t)_t), \mu((a_1^t)_t), \delta)$. This strategy profile is a δ-equilibrium by construction. In particular, the continuation play after the first ℓ periods constitutes a non-revealing δ-equilibrium, and the payoff to player 1 is tailored to make player 1 indifferent to the first ℓ periods' play.

All that remains is to show some finite-support $p \in \mathcal{I}(\mu_0)$ exists such that every μ in the support of p has $s_1 \in \operatorname{int} F_1^*(\mu)$. To that end, let us first show that F^* is a continuous and nonempty-, compact-valued correspondence. Since F^* is a Kakutani correspondence, we need only show that F^* is lower hemicontinuous. To that end, note that (because $\underline{u}_2$ is convex) the correspondence F^* is obtained as intersection of $F(\cdot)$ and $\tilde{F}(\cdot) = \{s \in \mathcal{R}^2 : s_1 \ge \underline{u}_1,\ s_2 \ge \underline{u}_2(\cdot)\} = \{s \in \mathcal{R}^2 : s_1 \ge \underline{u}_1,\ s_2 \ge \operatorname{vex} \underline{u}_2(\cdot)\}$. Because F and $\tilde{F}$ are convex-valued, and their intersection has a nonempty interior by hypothesis, lower hemicontinuity of F^* follows if F and $\tilde{F}$ are both lower hemicontinuous (Stokey et al., 1989, Exercise 3.12d). Since F is the convex hull of the correspondence formed by a finite number of continuous functions, F is also continuous (Aliprantis and Border, 2006, Theorem 17.37); in particular, F is lower hemicontinuous. That $\tilde{F}$ is lower hemicontinuous follows from the fact that $\underline{u}_2(\cdot)$ is continuous. Therefore, F^* is continuous and nonempty-, compact-valued, and so Berge's theorem gives that the functions $\min F_1^*, \max F_1^* : \Delta\Theta \to \mathcal{R}$ are continuous. Because $\Delta\Theta$ is compact and every $\mu \in \Delta\Theta$ has $\max F_1^*(\mu) > \min F_1^*(\mu)$ (because $F^*(\mu)$ has nonempty interior) some $\epsilon > 0$ then exists such that $\max F_1^*(\mu) - \min F_1^*(\mu) \ge 2\epsilon$ for all $\mu \in \Delta\Theta$. Making ϵ smaller if needed, we may further assume $s_1 \in [\underline{s}_1 + \epsilon, \overline{s}_1 - \epsilon]$. Now, the correspondence $F_1^\epsilon : \Delta\Theta \rightrightarrows \mathcal{R}$ given by $F_1^\epsilon(\mu) := [\min F_1^*(\mu) + \epsilon,\ \max F_1^*(\mu) - \epsilon]$ is a Kakutani correspondence by construction. Let S_1^ϵ denote the set of s_1' for which some $p \in \mathcal{I}(\mu_0)$ exists such that

[30] Specifically, the candidate δ-equilibrium payoff vector s only appears twice in the proof. First, at the beginning of the proof when choosing $s' \in \operatorname{int} F^{**}(\mu)$ and $e > 0$ such that (i) an open ball of radius $4e$ entered around s' is contained in $\operatorname{int} F^{**}(\mu)$ and (ii) $s_i - s_i' \ge 2e$ for each $i \in \{1,2\}$. Clearly, if $\tilde{s} \ge s$, then the same s' satisfies the two conditions. The second time s^μ appears is when checking that players would not deviate from on-path play. But if the players do not have the incentive to deviate under the on-path payoff of s, then they would not deviate under $\tilde{s} \ge s$ *a fortiori*.

Player's payoffs when $\theta = 0$: $(u_1(a),\, u_2(a, 0))$

	$a_2 = L$	R
$a_1 = T$	$4, 4$	$2, 0$
B	$2, 6$	$4, -8$

Player's payoffs when $\theta = 1$: $(u_1(a),\, u_2(a, 1))$

	$a_2 = L$	R
$a_1 = T$	$4, -4$	$2, 5$
B	$2, 0$	$4, 4$

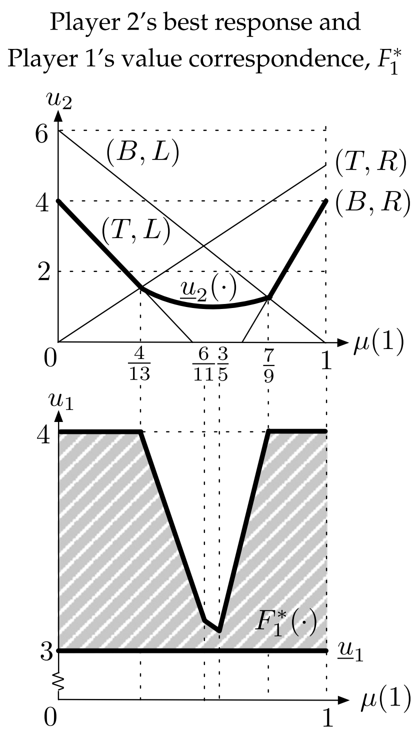

Fig. A.1. Example A.1.

$$p\left(\left\{\min F_1^*(\mu) + \epsilon \le s_1 \le \max F_1^*(\mu) - \epsilon\right\}\right) = 1.$$

Because Θ is finite, Theorem 17.2 in Rockafellar (1997) implies one can further take p to have finite support. Applying Theorem 1 (or 2) from Lipnowski and Ravid (2020) to the correspondences F_1^ϵ and $-F_1^\epsilon$, we obtain that the maximum and minimum of S_1^ϵ are $\overline{s}_1 - \epsilon$ and $\underline{s}_1 + \epsilon$, respectively. We can also apply Corollary 3 from Lipnowski and Ravid (2020) to F_1^ϵ to learn S_1^ϵ is convex. Therefore, $s_1 \in S_1^\epsilon$, and so some p with the desired properties exists. □

We provide a game that satisfies the conditions stated in Claim A.1 in the example below.

Example A.1. There are two states, $\Theta := \{0, 1\}$, and players 1 and 2 both have two actions $A_1 := \{T, B\}$ and $A_2 := \{L, R\}$, respectively. Fig. A.1 shows the payoffs associated with each action profile, player 2's expected payoffs associated with each action profile, player 2's maxmin payoff $\underline{u}_2(\cdot)$, as well as F_1^*. Observe that $\underline{u}_2(\cdot)$ is convex. One can also show that $F^*(\mu)$ has nonempty interior for all $\mu \in \Delta\Theta$. By Lemma 2, every $s_1 \in [3, 4]$ is an AMS-equilibrium $P1$-payoff, and Proposition 1 gives us that any such s_1 is also an equilibrium $P1$-payoff. Claim A.1 further gives that any $s_1 \in (3, 4)$ is a δ-equilibrium $P1$-payoff for sufficiently large discount rate δ. □

References

Aliprantis, Charalambos D., Border, Kim C., 2006. Infinite Dimensional Analysis: A Hitchhiker's Guide, 3 ed. Springer, Berlin, Heidelberg.
Aumann, Robert J., Shapley, Lloyd S., 1994. Long-term competition—a game-theoretic analysis. In: Megiddo, Nimrod (Ed.), Essays in Game Theory: In Honor of Michael Maschler. Springer, New York, NY, pp. 1–15. Chapter 1.
Aumann, Robert J., Maschler, Michael B., 1966. Game theoretic aspects of gradual disarmament. Report of the US Arms Control and Disarmament Agency 80. pp. 1–55.
Aumann, Robert J., Maschler, Michael B., 1968. Repeated games of incomplete information: the zero-sum extensive case. Mathematica, Inc, 37–116.
Aumann, Robert J., Maschler, Michael B., 1995. Repeated Games with Incomplete Information. MIT Press.
Aumann, Robert J., Hart, Sergiu, 1986. Bi-convexity and bi-martingales. Isr. J. Math. 54, 159–180.
Aumann, Robert J., Hart, Sergiu, 2003. Long cheap talk. Econometrica 71 (6), 1619–1660.
Aumann, Robert J., Maschler, Michael B., Stearns, Richard E., 1968. Repeated games of incomplete information: an approach to the non-zero-sum case. Report of the US Arms Control and Disarmament Agency ST-143. pp. 117–216.
Chakraborty, Archishman, Harbaugh, Rick, 2010. Persuasion by cheap talk. Am. Econ. Rev. 100, 2361–2382.
Cripps, Martin W., Thomas, Jonathan P., 2003. Some asymptotic results in discounted repeated games of one-sided incomplete information. Math. Oper. Res. 28 (3), 433–462.
Forges, Françoise, 1984. Note on nash equilibria in infinitely repeated games with incomplete information. Int. J. Game Theory 13 (3), 179–187.
Forges, Françoise, 1994. Non-zero sum repeated games and information transmission. In: Megiddo, Nimrod (Ed.), Essays in Game Theory: In Honor of Michael Maschler, 1 ed. Springer, New York, NY, pp. 65–96. Chapter 6.
Forges, Françoise, 2020. Games with incomplete information: from repetition to cheap talk and persuasion. Ann. Econ. Statist. 137, 3–30.
Fudenberg, Drew, Maskin, Eric, 1986. The folk theorem in repeated games with discounting or with incomplete information. Econometrica 54, 533–554.

Hart, Sergiu, 1985. Nonzero-sum two-person repeated games with incomplete information. Math. Oper. Res. 10 (1), 117–153.
Lipnowski, Elliot, Ravid, Doron, 2020. Cheap talk with transparent motives. Econometrica 88 (4), 1631–1660.
Mailath, George J., Samuleson, Larry, 2006. Repeated Games and Reputations: Long-Run Relationships. Oxford University Press.
Mertens, Jean-François, 1987. Repeated Games, vol. 2. American Mathematical Society, Providence, RI, pp. 1528–1577.
Mertens, Jean-François, Sorin, Sylvain, Zamir, Shmuel, 2015. Repeated Games. Econometric Society Monographs. Cambridge University Press.
Pęski, Marcin, 2008. Repeated games with incomplete information on one side. Theor. Econ. 3 (1), 29–84.
Pęski, Marcin, 2014. Repeated games with incomplete information and discounting. Theor. Econ. 9 (3), 651–694.
Rockafellar, R. Tyrrell, 1997. Convex Analysis. Princeton Landmarks in Mathematics and Physics. Princeton University Press.
Rubinstein, Ariel, 1979. Equilibrium in supergames with the overtaking criterion. J. Econ. Theory 21, 1–9.
Shalev, Jonathan, 1994. Nonzero-sum two-person repeated games with incomplete information and known-own payoffs. Games Econ. Behav. 7 (2), 246–259.
Simon, R.S., Spież, S., Toruńczyk, H., 1995. The existence of equilibria in certain games, separation for families of convex functions and a theorem of Borsuk-Ulam type. Isr. J. Math. 92, 1–21.
Stokey, Nancy L., Lucas, Robert E., Prescott, Edward C., 1989. Recursive Methods in Economic Dynamics. Harvard University Press.